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ALTERNATING CURRENTS:

AN ANALYTICAL AND GRAPHICAL TREATMENT FOR STUDENTS AND ENGINEERS.

BY

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AND

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PREFACE.

THE recent advances made in the utilization of alternating currents and alternating current apparatus of all descriptions have been of such importance that there are now many interested in this field of work who desire to become conversant with the underlying principles of the subject in order that they may become better equipped to undertake the vast engineering problems which are constantly arising. In its newness, the theory of alternating currents has been developed this way and that, added to here and there, so that it is necessary for one to stop and consider the basis from which certain conclusions are reached, and the logical sequence by which the results are attained. Although many of the problems which arise have been fully treated by various writers, the solutions have as a rule been limited in their application to certain special cases, and have for the most part been presented in a fragmentary manner. This lack of a clear and succinct treatment, sufficiently broad to be general in its application, has been strongly felt, and it is in order to meet this demand for definite information in regard to the fundamental principles governing the flow of variable or alternating currents that this work is now presented to the public.

The purpose has been to use such mathematical terms and analytical methods as make it possible for the dem-

onstrations to be exact and rigorous, and at the same time to express the results in such a way as to be perfectly intelligible to those who do not desire to follow the methods of proof, but are only interested in the conclusions reached.

There are some to whom graphical methods appeal more strongly than analytical processes, and the cases of simple circuits have accordingly been fully treated in both ways. The problems of divided circuits and networks of conductors yield the more readily to graphical treatment, inasmuch as analytical methods necessarily become cumbersome and involved and do not appeal directly to the senses. The subject is therefore capable of two natural divisions, the analytical, which constitutes Part I., and the graphical, which constitutes Part II.

In Part I. the discussion of circuits containing resistance and self-induction only is first taken up, and the first chapter contains the elementary principles necessary for the establishment of the equation of energy for such circuits. This equation is logically developed from the experiments of Coulomb, Faraday, Joule, and Ohm, upon which depends all the modern science of electricity. The treatment is based upon simple elementary ideas and is complete in itself, so that no previous knowledge of the theory of electricity and magnetism is requisite. Taking the equation of energy as a basis, in the following chapters the general solution for the current is obtained, after which the various particular cases are taken up, in which the electromotive force is assumed to vary as some definite function of the time. The solution for each particular case is derived independently from the differential equations and also from the general integral equation.

Inasmuch as the assumption of an harmonic electromotive force often approximates to the truth, a chapter

has been devoted to the discussion of harmonic functions in order that the solutions obtained under such an assumption may be the more clearly understood.

As is explained in the introductory chapter, the coefficient of self-induction L is considered constant, whereas this is only strictly true if the permeability of the surrounding medium is also constant. That this assumption is nearly correct is readily seen by noting the curves of magnetization for various commercial irons given by Prof. Ewing, and by Mr. Steinmetz and Mr. M. E. Thompson in this country, for it is not until a higher degree of magnetization is reached than is ordinarily met with in actual practice, that these curves deviate materially from a straight line.

After the completion of the treatment of circuits containing resistance and self-induction, the discussion of circuits containing resistance and capacity is taken up and developed in a similar manner from elementary principles. From the simple ideas of static charge, the meaning of potential and work is shown, leading up to the derivation of the equation of energy and electromotive forces for a circuit containing a condenser. Following the same plan as in the treatment of circuits containing self-induction, the general solution is first obtained, and we thus have the expression for the current and charge at any time for any impressed electromotive force whatsoever. Particular electromotive forces are then assumed, and the solutions for these cases are obtained from the general integral equation, and also independently by particular solutions.

The general case of circuits containing resistance, self-induction, and capacity is next taken up, and the same order of treatment is followed as in the discussion of circuits containing resistance and self-induction only, and resistance and capacity only. Now that the con-

denser, as well as its older brother, the transformer, is being applied to practical uses, the question of the action of a condenser in a circuit with self-induction becomes an important one, and the discussion of this case is given at length, the same method of giving particular cases after the general solution being followed as before. The case of oscillatory and non-oscillatory charge is treated at length as well as the corresponding case of discharge. In order that the effects caused by the variation of the constants of a circuit may be clearly understood, curves are drawn showing these effects for certain particular cases. The nature of the flow of current immediately after making a circuit is then investigated, and the results shown by plotting the instantaneous values for a particular case. The neutralizing effects of self-induction and capacity are next discussed, and the necessary conditions ascertained under which not only the instantaneous values of the current will be the same as though the self-induction and capacity were absent, but likewise the thermic and dynamometric effects.

The first part closes with an investigation of the nature of wave propagation in a conductor possessing self-induction and distributed capacity, a subject which assumes importance in submarine cables and in extended telephone circuits.

The results obtained by analytical processes too often fail to carry their full significance while in symbolic form, and for this reason it has been found advisable to give applications to concrete cases, and to draw curves illustrating the points involved. In order that the full significance of the results may be grasped, the values of the quantities used in these numerical examples have in all cases been given, so that the curves plotted show not only the general nature of the relations between the various quantities, but also the value of these quantities

in the particular cases assumed. The advantage of this is especially shown in the discussion of the effects of the variation of the constants in a circuit containing resistance, self-induction and capacity, for it is the illustrations which here bring out the true significance of the effects.

In Part II. the same order is followed as in Part I. The graphical method of treating problems of simple circuits containing resistance and self-induction is first fully established from the analytical results obtained in Part I., and is then extended to problems arising in the case of combination circuits. Problems arising in the case of simple and combination circuits containing resistance and capacity but no self-induction are then solved, and finally the general case of circuits containing resistance, self-induction and capacity is taken up, and the graphical solutions given for series, parallel and combined circuits.

The graphical methods are rigorously proved by the analytical solutions obtained in the earlier part of the book, but the development is such that those who do not follow through the analytical proof may readily apply these graphical methods to the solution of practical problems.

In order to avoid ambiguity, the same symbols are used throughout with the same signification, and a list of symbols used, together with their meanings, is given in an appendix.

There have been many valuable papers on subjects relating to alternating currents, among others those by Dr. Duncan and Prof. Ryan in this country, and by Prof. Ayrton, Dr. Sumpner, Dr. Fleming, and Mr. Blakesley in England, and the electrical public has gained much information from the excellent works of the last two writers. The subject has not, however, been hitherto

developed in the way followed in the succeeding pages, and it is in order to meet the demand for a logical treatment of the theory of alternating currents that this book has been prepared.

Much of the matter here contained has already been given by the writers in various papers, some of which originally appeared as a series of articles in the *Electrical World*, and others in the *London Electrician*, the *Philosophical Magazine*, the *American Journal of Science*, and the *Transactions of the American Institute of Electrical Engineers*. We have been permitted to use some of the cuts from the latter, for which courtesy we desire to extend our thanks.

The matter contained in the second part now appears for the first time, with the exception of the method for obtaining the equivalent resistance, self-induction and capacity of parallel circuits, which was first given in the *Philosophical Magazine*.

In all cases these papers have been carefully revised and rewritten, and in many cases amplified to suit the requirements of the book.

CORNELL UNIVERSITY, ITHACA, N. Y.,

PREFACE TO FIFTH EDITION.

IN this edition, some minor changes have been made and a few notes added, with no material alteration in either the subject matter or the arrangement.

ITHACA, N. Y., 1909.

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PART I.

ANALYTICAL TREATMENT.

CHAPTER I.

INTRODUCTORY TO CIRCUITS CONTAINING RESISTANCE AND SELF INDUCTION.

CONTENTS:—Magnet. Lines of force. Field of force. Pole. North pole. Like poles repel, unlike attract. Unit pole. Law of attraction. Intensity of a field of force. Uniform field. Unit line of force. Unit pole has 4π lines of force. Induction. Current develops a field. Unit current. Number of lines proportional to current. Self-induction. E. M. F. Ohm's law. Quantity. Quantity definite for definite change in lines. Joule's law. Energy dissipated in heat. Total energy imparted to a circuit. Energy expended in field. Equation of energy. Equation of E. M. F.'s.

IN order that circuits containing resistance and self-induction may be properly discussed, a brief review will first be given of the elementary theory of magnetism, the nature of the magnetic field, and the relation between a current of electricity and magnetism. Those well-known elements of the subject will be presented which enable us to obtain expressions for the energy imparted to a circuit, the energy dissipated in heat, and the energy expended in the magnetic field, and finally to establish the equation of energy and the equation of electromotive forces for circuits with resistance and self-induction.

If a needle is magnetized uniformly in the direction of its length and placed in iron filings, the filings are attracted to the ends of the needle and become attached thereto in clusters. The attractive power of the magnet-

ized needle is apparently concentrated at the ends, which are called *poles*. The filings in the space around the magnet tend to gather in lines, called *lines of force*, extending from one pole of the needle to the other. Thus the magnet is seen to be surrounded by a *field of force*, in which the lines indicate the direction of the force at any point of the field. When a compass-needle is placed in the field, it always assumes a definite position, tangent to the line of force passing through that point. The earth acts like a huge magnet, producing a magnetic field in which the lines of force have a direction nearly north and south. A magnetized needle freely suspended in the earth's magnetic field assumes a definite position tangent to the earth's lines of force. This position is usually nearly in the geographical meridian, the magnet having one pole toward the north and the other toward the south. The pole that is toward the north is called the positive pole, marked +; and the opposite pole the negative, marked —. When magnetic poles are brought near one another there is found to be either an attraction or a repulsion between them, and two poles which have the same sign tend to repel one another, while two poles of opposite sign tend to attract one another.

The definition of a *unit magnetic pole* would therefore naturally be: a magnetic pole which exerts a force of one dyne* upon another equal pole at a distance of one centi-

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* Our knowledge of the physical universe is obtained from our perception of matter in its relations to time and space; and physical measurements are, accordingly, measurements of mass, length, and time. Any quantity can be expressed in terms of these three, and the units in which the quantity is measured can be expressed in the three *fundamental* units of *length*, *mass*, and *time*. The fundamental units commonly used to measure length, mass, and time are the *centimetre*, *gramme*, and *second*; these are arbitrarily selected, and give rise to the C. G. S. system of units. All other units are readily obtained from these and are called *derived* units. The *velocity* of a body moving uniformly is the space passed over in a unit time. For a body having a variable motion, the velocity is equal to an

metre. Such a magnetic pole as this just defined forms the foundation upon which is based the whole system of electromagnetic units, those of current, electromotive force, etc.; and it therefore deserves attention.

This definition depends upon the exact measurement of the distance between two poles. But in reality magnetic poles have finite dimensions and it is necessary to determine the mean distance between them. The distance taken is that between two points so situated that the action between the two poles would be the same if they were concentrated at these two points. We therefore think of a magnetic pole as concentrated at a point. This conception is no more strained than the conception of centre of gravitative attraction of a body, where we consider the whole mass of the body as concentrated at a point.

The length of a compound pendulum is measured in a similar way, by considering that the mass of the pendulum is concentrated at such a point that the time of oscillation is not changed.

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element of the distance ds, divided by the time dt, in which the distance ds is traversed; that is, velocity equals the rate of change of length with respect to time, $v = \frac{ds}{dt}$. In the C. G. S. system velocity is measured in centimetres per second. The acceleration of the body is the rate at which the velocity is changing; that is, $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$. In the C. G. S. system acceleration is measured in centimetres per second.

By Newton's first law, every body continues in a state of rest, or of uniform motion in a straight line, except in so far as it may be compelled by impressed forces to change that state. *Force* may be defined as that which causes or tends to cause a change in the velocity of a body. The unit of force is that force which causes a unit change in velocity of a unit mass in unit time, that is, produces unit acceleration. In the C. G. S. system the unit of force is the *dyne*, and is the force which, when acting for one second, will give a mass of one gramme a velocity of one centimetre per second.

LAW OF ATTRACTION.

The law of the action between magnetic poles, as experimentally determined by Coulomb, is that the attraction or repulsion between two poles is inversely as the square of the distance between them, and directly as the product of their strengths; that is,

$$F \propto \frac{m\,m'}{r^2},$$

where m and m' are the strengths of the poles, that is, the number of unit poles to which each is equivalent, and r the distance between them.

A unit pole being as previously defined, the sign of variation may be changed for one of equality if the distance r is measured in centimetres and the force F is measured in dynes. The force between two magnetic poles is then

$$F = \frac{m\,m'}{r^2}.$$

When the poles considered have the same sign, and are both north poles or both south poles, the product $m\,m'$ is positive, and a force of repulsion has the positive sign. Similarly, a force of attraction has a negative sign.

INTENSITY OF A FIELD OF FORCE.

The strength of a magnetic field of force at any point is measured by its action on a unit positive magnetic pole at that point.

If we could place a free magnetic pole in a magnetic field, it would always be urged in a certain direction; and, if free to move, would actually move in this direction. The direction in which a positive pole would be urged is called the positive direction of the line of force which

passes through the pole. The force with which a unit positive pole would be urged at any point of a magnetic field is the strength of the field at that point, and is usually denoted by **H**.

Usually it is found that **H** varies at different points in the field; but if **H** has the same value at every point, both in magnitude and in direction, the field is said to be uniform.

If the uniform field be one of *unit* intensity, then $\mathbf{H} = 1$, and there is said to be *one line of force per square centimetre*; and when the intensity is **H**, there are **H** lines of force per square centimetre. Thus the intensity of a magnetic field is thought of as being determined by the number of lines which pass through one square centimetre of a surface normal to the direction of the lines of force.

As an example, by the definition of a unit pole the intensity of the field **H** is unity at a distance of one centimetre from the pole. If a sphere be described about the unit pole as a centre, having a radius of one centimetre, there is consequently one line of force passing through the surface of the sphere for every square centimetre. As the surface of the sphere contains $4\,\pi$ square centimetres, there are in all $4\,\pi$ lines of force that emanate from a unit pole, and $4\,\pi\,m$ lines from a pole whose strength is m.

INDUCTION.

The number of lines of force in air is the same as the number of lines of magnetizing force. In a magnetic substance, such as iron, the number of lines of force is greatly increased, and they are then called lines of magnetization, or lines of induction.

The number of lines of induction N, passing through any area, is called the total magnetic induction through this area. The total number of lines of force per square

centimetre of area normal to these lines is called the induction per square centimetre, or simply the *induction*, **B**. In a non-magnetic medium the induction **B** is equal to the magnetizing force, **H**. In a magnetic medium, such as iron, the magnetizing force produces an induction **B** greater than **H**. The ratio of the induction to the magnetizing force is called the *permeability*, μ; that is, $\mu = \frac{\mathbf{B}}{\mathbf{H}}$.

A current of electricity flowing in a circuit always produces a magnetic field in the surrounding region. The lines of force which constitute this field are always closed lines which encircle the conductor. The total number of lines passing through the area bounded by a closed electric circuit is the total magnetic induction of the circuit. As the current is increased in strength, the intensity of the magnetic field at every point is increased, and if there is no magnetic substance in the region, the intensity of the field is increased in direct proportion to the strength of current.

A *unit current* is defined in terms of the intensity of the magnetic field which it generates. A unit current is that which, flowing in a circuit of one centimetre radius, acts on a unit magnetic pole, placed at the centre, with a force of one dyne per centimetre length of the circumference. This is a unit of current in the C. G. S. system. Each unit length of conductor is acted upon by the unit pole, placed at the centre, with a force of one dyne, which is the same force as that by which a unit pole would be acted upon when substituted for a unit length of the conductor at the same distance. The practical unit of current, the *ampere*, is one-tenth of the C. G. S. unit.

NUMBER OF LINES PROPORTIONAL TO CURRENT.

We have seen that when a current flows through a closed circuit a field is set up consisting of a definite num-

ber of lines threading the circuit. If there is no magnetic substance in the vicinity, that is if the *permeability* of the surrounding region be constant, the number of lines produced by a current in a circuit is directly proportional to the current, and any change in the current produces a proportional change in the number of lines threading the circuit.

This may be expressed $N \propto i$, and $\frac{dN}{dt} \propto \frac{di}{dt}$. Since N varies as i, we may say

$$N = L\,i,$$

(1) and consequently $$\frac{dN}{dt} = L\frac{di}{dt}.$$

The coefficient L is called the *coefficient of self-induction*, and is defined by the equation as the ratio of the total induction threading a circuit to the current producing it. When the current is unity, the coefficient of self-induction is equal to the number of lines produced by the current. If the permeability of the medium surrounding the conductor is constant, this will be the value of L for all values of the current, and L will be constant. Unless a high degree of magnetization is reached, L is approximately a constant for a given circuit, and will hereafter be so considered.

FARADAY'S LAW OF ELECTROMOTIVE FORCE.

When a conductor is moved in a magnetic field so as to cut lines of force an electromotive force is produced in the conductor. Faraday showed as the result of his researches that this E. M. F. produced is directly proportional to the rate of cutting the lines of force, and is in a direction at right angles to the direction of motion and also to the direction of the lines of force. He further showed that, if the magnetic induction through any closed circuit be varied by any means, an E. M. F. is developed in the

circuit proportional at any instant to the rate of change (decrease) of the magnetic induction at that instant. In the C. G. S. system of units this experimental law may be expressed by the equation

$$(2) \qquad e = -\frac{dN}{dt},$$

where e denotes the E. M. F. developed and N the magnetic induction of the circuit. This means that a C. G. S. unit E. M. F. is developed when there is a change in the induction of the circuit at the rate of one line per second. The negative sign indicates that the E. M. F. is induced in such a direction as to oppose the change in the number of lines threading the circuit. The practical unit of E. M. F., the *volt*, is 10^8 times the C. G. S. unit just defined.

OHM'S LAW.

An electromotive force impressed upon a closed circuit causes a current to flow which depends upon the resistance of the circuit. Ohm first showed, and others have since verified to a high degree of accuracy, that with a constant E. M. F., the current is directly proportional to the E. M. F. and inversely proportional to the resistance of the circuit.

Ohm's law may be expressed

$$I \propto \frac{E}{R},$$

where I denotes current; E, electromotive force; and R, resistance. Since E. M. F. and current are already independently defined, the unit of resistance is naturally taken to be that resistance which allows a unit current to flow in a circuit having a unit impressed E. M. F. Ohm's law may then be expressed

$$I = \frac{E}{R}.$$

From the relation of the practical units of E. M. F. and current, the volt and the ampere, to the corresponding C. G. S. units, it follows that the practical unit of resistance, the *ohm*, is 10^9 times the C. G. S. unit.

QUANTITY.

A unit *quantity* of electricity is said to flow in a circuit when unit current flows for one second. When a current I flows in a circuit for t seconds, a quantity It will flow. And in a short interval of time dt, a quantity $i\,dt$ will flow, which is represented by dq, thus:

$$\frac{dq}{dt} = i,$$

q representing quantity.

We have seen that for a constant electromotive force, by Ohm's law the current equals the E. M. F. divided by the resistance. During a short interval of time dt, any E. M. F. may be considered constant, and we may write

$$i = \frac{e}{R}, \text{ during the time } dt.$$

The capital letters E, I, and Q will be used to denote a constant electromotive force, current, or charge. When these are variable the small letters e, i, and q will be used.

When a closed conductor is moved from one position to another in a magnetic field, so as to cause the number of lines of force included by the circuit to change from one value N_1 to another value N_2, it will be found that the quantity of electricity which flows in the circuit is always a definite amount, being equal to the change in the number of lines $N_2 - N_1$, divided by the resistance of the circuit, and is entirely independent of the manner of the change, and of the time occupied in making the change.

This will be evident when we remember Faraday's law, $e = -\frac{dN}{dt}$, and consider that the only E. M. F. acting in the circuit during the motion of the conductor is this $-\frac{dN}{dt}$. Hence the following relations are true:

$$\frac{dq}{dt} = i = \frac{e}{R} = -\frac{1}{R}\frac{dN}{dt}.$$

(3) Whence $Q = \frac{N_1 - N_2}{R}$.

Here Q denotes the sum of all the small quantities, or the total quantity of electricity flowing through the circuit during the motion of the conductor, and is seen to be equal to the change of the induction through the circuit divided by the resistance of the circuit, as stated above.

The earth inductor is a good example of an instrument which depends for its use upon the principle just stated. When a ballistic galvanometer is connected with an earth inductor, the throw of the galvanometer is proportional to the total change in the number of lines of force included by the earth inductor coil as it turns from one position to another, provided the needle does not start to swing until the whole quantity of electricity has flowed through the galvanometer.

JOULE'S LAW.

The fourth and last great experimental law to be mentioned is the discovery by Joule that the heat liberated by a conductor carrying a current of electricity is strictly proportional to the product of the square of the current-strength and the resistance of the conductor.

Now for the first time we have a means of telling how much power is required to send a current of any desired strength through a conductor, and we always expect to find

some source for the supply of energy when we see a current flowing through a conductor.

The elementary principles, already given, upon which the system of electromagnetic units is based, are deduced from the experimental researches of Coulomb, Faraday, and Ohm. When a current flows through a conductor there is always heat liberated in the conductor and accordingly a dissipation of energy. It therefore requires a certain amount of power to send a current through a conductor. The exact amount of this heating effect was first determined by Joule. The results of his experiments show that the energy liberated per second in the form of heat in a conductor carrying a current of electricity is strictly proportional to the product of the square of the current-strength and the resistance of the conductor. Joule's law may be expressed

$$W \propto I^2 R,$$

where W represents the energy expended per second.

The energy expended in the time t, during which the current is constant, is $I^2 R t$. If the current be a variable i, it may be considered constant for the time dt, and so in the time dt

(4) Energy dissipated in heat is $dW = i^2 R \, dt$.

When *all* the energy given to the circuit is expended in heat, that is when there is no counter E. M. F. of any kind and the current is constant, IR may be replaced by E, according to Ohm's law, and the energy expended in the time t may be written

$$W \propto EIt.$$

This becomes more definite in the units already discussed. If a conductor carrying a current I be placed at right angles to the lines of force in a uniform field of

intensity $\mathbf{H}$, each unit of length will be acted upon with a force $\mathbf{H}\,I$. If l be the length of the conductor, the force will be $l\,\mathbf{H}\,I$. When moved with a velocity v against this force, work will be performed at the rate of $l\,\mathbf{H}\,I v$ ergs per second, or in the time t the work done is

$$W = l\,\mathbf{H}\,I v t.$$

This must be equal to the rate at which work is done in generating a current I, by moving the conductor through the field. The conductor, when moving with a velocity v, cuts $l\,\mathbf{H}\,v$ lines per second, and so produces an E. M. F.

$$E = l\,\mathbf{H}\,v.$$

Substituting above, we see that the amount of energy expended in a circuit is equal to the product of the current, electromotive force, and time,

$$W = E I t.$$

This is seen to be equivalent to Joule's law above and is equally true for C. G. S. and for practical units. In the C. G. S. system, energy is measured in *ergs* and the equation expresses the fact that energy in ergs is equal to the product of current, E. M. F., and time in C. G. S. units. The practical unit of energy is the *joule* and is so defined that the equation $W = E I t$, true in ergs and other C. G. S. units, shall be also true in practical units—the joule, the volt, and the ampere. The equation is then interpreted as meaning that energy in joules is equal to the product of current, E. M. F., and time in amperes, volts, and seconds. From the relation already given between the ampere and volt and their corresponding C. G. S. units, the joule equals 10^7 times the C. G. S. unit the erg.

The rate of work is in electrical terms expressed in watts: one watt equals one joule per second. The common

English unit of rate of work is the horse-power: one horse-power equals 745.9 watts.

The rate at which energy is imparted to a circuit multiplied by the time is the total energy imparted during that time. If there is a variable E. M. F., e, from any source whatever given to a circuit, and a current i flows, the energy imparted to the circuit in the time dt from the source of this E. M. F. is the product $e\,i\,dt$. Thus:

(5) Energy imparted to a circuit $= e\,i\,dt$.

This enables us to ascertain the energy possessed by a magnetic field. By Faraday's law, when the magnetic induction through any closed circuit changes from any cause whatsoever, there is always an electromotive force given to the circuit which is equal to

$$e = -\frac{dN}{dt} = -L\frac{di}{dt}.$$

This E. M. F. is due to the existence of the magnetic field. In creating the field, an equal and opposite E. M. F., $L\frac{di}{dt}$, is necessary to drive the current. The work which this force does is equal to the product of the force, the current which flows in the circuit, and the time dt; as explained above. The change in the energy possessed by a magnetic field in the time dt is, therefore,

$$i\frac{dN}{dt}dt = L\,i\frac{di}{dt}dt.$$

(6) Energy expended in the magnetic field $= L\,i\frac{di}{dt}dt$.

The change in the induction through any circuit may be due to any external cause, as moving magnets, or it may be due to a change in the current itself. When the change is due to a change in the current, an increase in the

strength of the current increases the energy of the magnetic field; and positive work is done by the current in creating the field. When the current decreases, the energy of the field decreases, and negative work is done by the current on the field; for, when the current decreases with the time, $\frac{di}{dt}$ is negative. To say that the current is doing negative work is equivalent to saying that the magnetic field in decreasing is imparting energy to the circuit. Thus we see that the energy may be stored up in a magnetic field, and that this is not dissipated, but is returned to the circuit when the field is diminished in strength.

To find the expression for the total energy of a magnetic field which is due to a current i flowing in a circuit, we need merely find the sum of all the small quantities of energy imparted to the field as the current is increased from zero to its final value I: this is found to be

$$(7) \qquad \int_0^I L\,i\,di = \tfrac{1}{2}\,L\,I^2.$$

THE EQUATION OF ENERGY.

If e represents the impressed E. M. F. given to a circuit which has a resistance R and a coefficient of self-induction L, we have seen [equation (5)] that the total energy given to the circuit from the source is $e\,i\,dt$.

A part of this energy supplied is dissipated in heating the conductor, and in the time dt is equal to $R\,i^2\,dt$ [equation (4)]. A second part is stored up in the magnetic field, and in the time dt is equal to $L\,i\,\frac{di}{dt}\,dt$ [equation (6)]. These two ways are the only ones in which the energy of the source is used, under the hypothesis made that the circuit contains no statical capacity or counter electromotive force

of any kind other than that due to the field, but only contains a resistance R and a self-induction L.

By the principle of the conservation of energy we may, therefore, say that the energy supplied to the circuit is the sum of the energy dissipated in heat and the energy expended on the field.

We have, therefore, the equation of energy:

$$(8) \qquad e\,i\,dt = R\,i^2\,dt + L\,i\frac{di}{dt}dt.$$

When each member of the equation of energy is divided by $i\,dt$, we obtain

$$(9) \qquad e = R\,i + L\frac{di}{dt}.$$

This is an equation of electromotive forces: e is the E. M. F. of the source impressed upon the circuit, $R\,i$ the E. M. F. necessary to overcome the ohmic resistance, and $L\frac{di}{dt}$ the E. M. F. equal to the E. M. F. of self-induction.

NOTE.—The number of lines or the total induction threading a circuit, when the circuit consists of a coil of several turns, is equal to the number of lines which pass through the coil as a whole, multiplied by the number of turns. One line passing through a coil of s turns actually threads the circuit s times; thus, if 3,000 lines pass through a coil of 50 turns, the total induction of the circuit is $N = 3{,}000 \times 50 = 150{,}000$. The explanations on p. 23 *et seq.* and equations (1), (2), and (3) are to be thus understood.

The coefficient of self-induction is defined in terms of the counter-electromotive force of self-induction as follows: $e = L\frac{di}{dt}$; hence, the coefficient of self-induction (or inductance) is the ratio of the counter-electromotive force of self-induction to the time-rate of change of the current producing it. The unit of self-induction, the henry, as defined by the International Electrical Congress, Chicago, 1893, is the self-induction of a circuit when the electromotive force induced in the circuit is one volt, while the inducing current varies at the rate of one ampere per second.

CHAPTER II.

ON HARMONIC FUNCTIONS.

CONTENTS :—Harmonic E. M. F. often assumed. Simple harmonic motion. Amplitude. Period. Angular velocity. Frequency. Epoch. Phase. Lag. Advance. Graphical representation of simple harmonic functions. Average value of ordinates of sine-curve. Value of mean square of ordinates of sine-curve. Periodic functions composed of several simple sine-functions of like period,—of unlike period. Fourier's theorem.

IF a conductor revolves with uniform velocity about some fixed axis in a uniform field, the rate at which it cuts the lines of force is different at different parts of the revolution and varies directly as the sine of the angle of rotation. The electromotive force set up in the conductor at any instant is numerically equal to the rate of cutting lines at that instant and is accordingly a sine-function of the angle of rotation and, since the rotation is uniform, a sine-function of the time. Inasmuch as the assumption of such an electromotive force often closely approximates to the truth, and since, as will be shown later, any electromotive force whatever may be expressed as a sum of terms each of which is a sine-function of the time, it has been found convenient to express electromotive forces in terms of sines.

In order that sine-functions may be clearly understood when used in the following chapters, it is considered advisable to digress and devote the present chapter to the discussion of harmonic or sine-functions.

HARMONIC MOTION.

If a point moves uniformly around the circumference of a circle, the motion of the projection of that point upon any fixed diameter is said to be *harmonic*. The radius of the circle is called the *amplitude* of the motion, and is designated by a. The time T of making one complete revolution is called the *period*. Positive rotation will be considered as counter-clockwise.

If a uniformly revolving radius of a circle is projected upon any fixed diameter, its projection is said to vary harmonically. The maximum value of this projection is the amplitude, or radius of the circle. This is represented in Fig. 1. P is a point moving uniformly about the centre

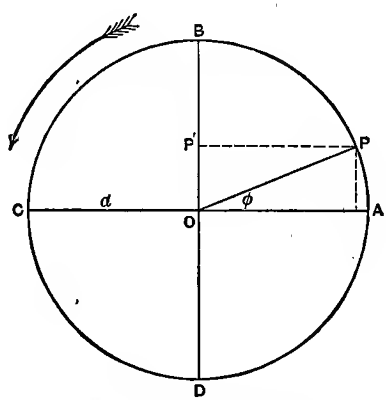

FIG. 1.—HARMONIC MOTION.

O, and $\overline{OP'}$ is the projection of the radius $\overline{OP}$ upon the fixed diameter $\overline{BD}$. When $\overline{OP}$ is in the position $\overline{OA}$ at right angles to $\overline{BD}$, the projection $\overline{OP'}$ is zero; and when $\overline{OP}$ is in the position $\overline{OB}$, the projection $\overline{OP'}$ has its maximum value and is equal to the radius $\overline{OP}$. The projection is again zero at $\overline{OC}$, and a negative maximum at $\overline{OD}$.

The *angular velocity* of the point P is denoted by ω, and at any point is $\omega = \frac{d\phi}{dt}$. Since the motion of the point is uniform, $\omega = \frac{\phi}{t}$, or $\omega t = \phi$, where ϕ is the angle described in the time t. As the time occupied in describing a circumference is T, the uniform velocity $\omega = \frac{2\pi}{T}$; hence $\phi = \frac{2\pi}{T}t$. The second is taken as the unit of time. The number of revolutions made by the moving point P, in one second is $\frac{1}{T}$ and is called the *periodicity* or *frequency*, often denoted by n. The frequency is the reciprocal of the period, i.e., $n = \frac{1}{T}$. It is evident that the angular velocity may be expressed in terms of the frequency; thus, $\omega = 2\pi n$, and therefore, $\phi = 2\pi nt$.

If we begin to count the time from the point A (Fig. 1), where the projection of $\overline{OP'}$ is zero, denoting $\overline{OP'}$ by y, we have at any time

$$y = a \sin \phi = a \sin \omega t,$$

where a denotes the amplitude and ϕ the angle described in the time t; y is an harmonic or sine-function of the angle ϕ or the time t.

Suppose that the time is counted from some point Q, Fig. 2, other than the point A at which the projection of $\overline{OP}$ is zero. There is an angle θ, called the *angle of epoch*, between the point from which time is reckoned and the point at which the projection of the radius is zero. The time in which this angle is described is called simply the *epoch*. As before, the angle ϕ is that described in the time t. The angle $(\phi + \theta)$, through which the point P has revolved from the point A where the projection of $\overline{OP}$ is zero, is called the *angle of phase* or briefly the *phase*. More strictly

defined, the phase is the ratio of the arc PA to the circumference of the circle.

If we denote by y the projection of $\overline{OP}$ upon $\overline{BD}$, and count time from Q,

$$(10) \qquad y = a \sin(\phi + \theta) = a \sin(\omega t + \theta).$$

When θ is positive,—that is when it is in the positive or counter-clockwise direction from A, as in Fig. 2,—it is often called the *angle of advance.* When θ is negative,—and the

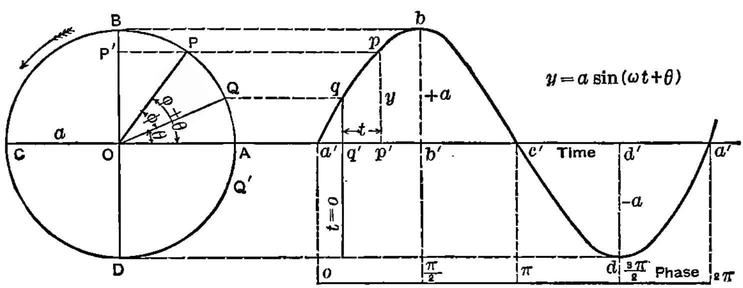

FIG. 2.—SIMPLE SINE-CURVE.

time is counted from some point Q' at a distance θ in the negative or clockwise direction from A,—it is often called the *angle of lag.* When the angle of phase is zero, $\overline{OP}$ coincides with $\overline{OA}$ and the projection $y = 0$. When the phase is 90°, the projection is a maximum, and $y = + a$. At 180°, again, $y = 0$; and at 270°, $y = - a$, a maximum in the negative direction. This cycle is repeated every revolution.

In the equation $y = a \sin(\omega t + \theta)$, the amplitude, a, angular velocity, ω, and angle of epoch, θ, are constants, and the variable y is said to be expressed as a simple sine-function of the variable t. The time t is directly proportional to the angular distance passed through. A variable whose value at any time can be expressed as a constant multiplied by the sine of an angle changing uniformly with

the time, is called a simple sine-function, or simple harmonic function of the time.

In Fig. 2, y is plotted as a sine-function of t. At any time, t, when the revolving point has the position P, y has a value $\overline{OP'}$.

Angle of epoch $= AOQ = \theta$.
Time of epoch $= a'q'$.
Angle described in time $t = QOP = \phi = \omega t$.
Angle of phase $= \phi +$ (angle of epoch) $= \phi + \theta$.
Time of phase $= t +$ (time of epoch) $= t + a'q'$.
Amplitude $= \overline{OA} = \overline{OB} = a$.

When the term "harmonic function" or "sine-function" or "sine-curve" is used, such a function or curve as shown in Fig. 2 is meant.

TO FIND THE AVERAGE VALUE OF THE ORDINATE OF A SINE-CURVE.

A sine-curve repeats itself symmetrically and the average ordinate for the whole period is, therefore, algebraically zero, as it is negative and positive alternately for equal intervals of time. We can, however, obtain the average value for one half a period and, since the second half is a repetition of the first half with sign reversed, this will give the arithmetical average value for the whole period.

The average ordinate is equal to the sum of all the vertical elements of area divided by their number, or, what is the same thing, it is equal to the area included between the curve and the axis of abscissæ, divided by the base. For half a period the limits are 0 and π, so the

$$\text{Average Ordinate} = \frac{\int_0^\pi y\,dx}{\int_0^\pi dx} = \frac{1}{\pi}\int_0^\pi y\,dx.$$

But for a sine-curve, $y = a \sin x$; therefore

$$\text{Average Ordinate} = \frac{a}{\pi}\int_0^\pi \sin\ x\,dx = \frac{a}{\pi}\Big[_0^\pi - \cos x = \frac{2a}{\pi}.$$

But a is the maximum ordinate, and $\frac{2}{\pi} = .6369$; so we may write

(11) $$\frac{\text{Average Ordinate}}{\text{Maximum Ordinate}} = .6369,$$

which determines the value of the average ordinate.

TO FIND THE VALUE OF THE MEAN SQUARE OF THE ORDINATES OF A SINE-CURVE.

Although it is often useful to know the value of the average ordinate of a sine-curve, it is more often desirable to know the value of the mean square of the ordinates, or the square root of the mean square. Since the square of an ordinate is positive irrespective of the sign of the ordinate, we can find the mean square of the ordinates by integrating for the whole and not for half a period as was necessary in finding the average ordinate.

$$\text{Mean Square of } y = \frac{\int_0^{2\pi} y^2\,dx}{\int_0^{2\pi} dx} = \frac{a^2}{2\pi}\int_0^{2\pi} \sin^2 x\,dx.$$

But $\sin^2 x = \frac{1}{2} - \frac{\cos 2x}{2}$. Therefore

$$\int_0^{2\pi} \sin^2 x\,dx = \Big[_0^{2\pi} \frac{x}{2} - \frac{\sin 2x}{4} = \pi.$$

Substituting π for the integral above, we have

(12) $$\text{Mean Square of } y = \frac{a^2}{2\pi} \times \pi = \frac{a^2}{2}.$$

The square root of the mean square of the ordinates is, therefore,

$$\frac{a}{\sqrt{2}} = .707a.$$

This means that the square root of the mean square of the instantaneous values of y, which varies harmonically with the time, is equal to .707 of the maximum ordinate.

The square root of the mean square of the instantaneous values of a variable current or electromotive force is called the *virtual* current or electromotive force and is equal to .707 times the maximum value of the current or electromotive force. [*Virtual* value is also called *effective* value.]

Inasmuch as the heating and dynamometer effects of any current depend directly upon its mean square value, this virtual value is of much more importance than the average value in the measurement of an alternating current.

PERIODIC FUNCTIONS COMPOSED OF SEVERAL SIMPLE SINE-FUNCTIONS.

A single-valued function is one which has but one value at any one point of time, as represented in Fig. 3. A multiple-valued function is one which may have more than one value at one point of time, as represented in Fig. 4. A periodic function is one which repeats itself after a definite time or period. If any number of simple sine-functions of the same period be added, the resultant sum will be a

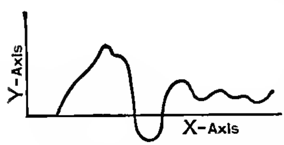

FIG. 3.—SINGLE-VALUED FUNCTION.

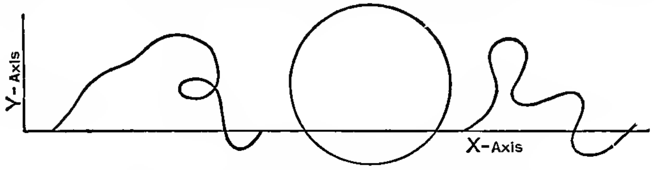

FIG. 4.—MULTIPLE-VALUED FUNCTION.

simple sine-function of the same period. This is rigorously

shown in Chap. XIV., Part II., for the addition of two simple sine-functions, as illustrated in Fig. 47; and it is evident that, if true for the addition of two, it is true for the ad-

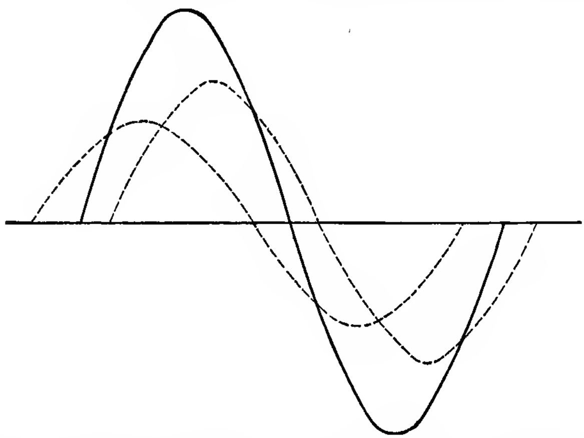

FIG. 5.—ADDITION OF SIMPLE HARMONIC CURVES OF SAME PERIOD.

dition of any number of simple sine-functions. An example of the addition of two simple sine-functions of the same

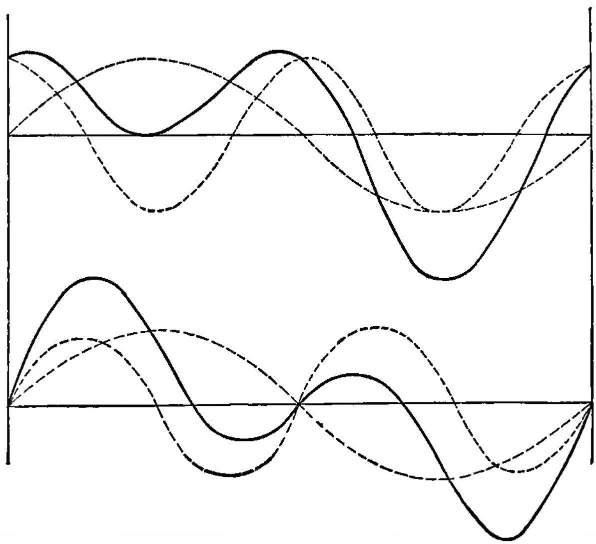

FIG. 6.—ADDITION OF SIMPLE HARMONIC CURVES OF DIFFERENT PERIODS.

period is shown in Fig. 5. The resultant curve, represented by the heavy line, is likewise a sine-curve.

If a number of simple sine-functions of periods which are different but commensurable, are added together, the resultant sum is a function which is periodic, but not harmonic, with a period equal to the least-common-multiple of the periods of the several component sine-functions. The two heavy curves in Fig. 6 are obtained by adding two simple sine-curves of the same amplitude and with periods in the ratio 1 : 2. The equation for the lower heavy curve is

$$y = a \sin \omega t + a \sin 2 \omega t,$$

the two component curves, shown by dotted lines, being zero at the start. The upper curve has the equation

$$y = a \sin \omega t + a \sin \left(2 \omega t + \frac{\pi}{2}\right),$$

the component dotted curves never being zero at the same time.

The addition of two sine-curves with different amplitudes

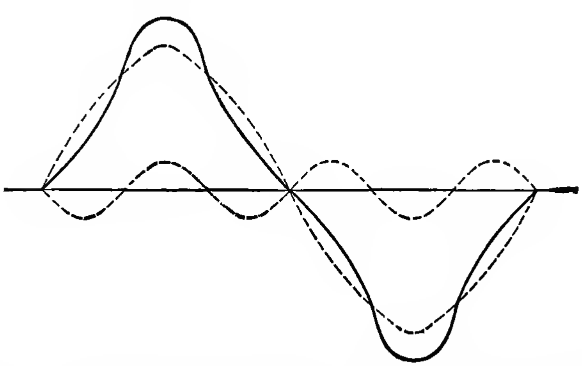

FIG 7.—ADDITION OF SIMPLE HARMONIC CURVES OF DIFFERENT PERIODS.

and with periods in the ratio 1 : 3 is illustrated in Figs. 7 and 8. The component curves in Fig. 7 have no phase

difference at the start and the resultant curve represents the equation

$$y = a \sin \omega t - b \sin 3 \omega t.$$

The curve in Fig. 8 represents the equation

$$y = a \sin \omega t + b \sin (3 \omega t + \theta).$$

By adding a number of component simple sine-curves with different periods and amplitudes, resultant periodic

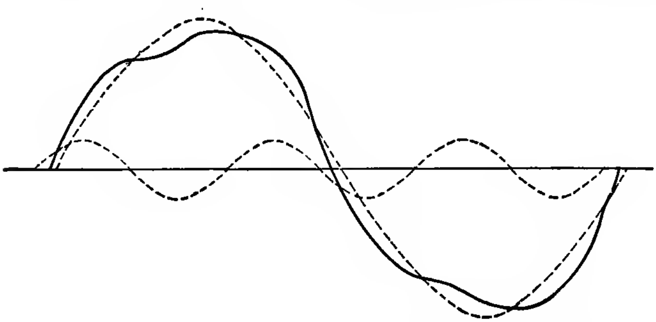

FIG. 8.—ADDITION OF SIMPLE HARMONIC CURVES OF DIFFERENT PERIODS.

curves of all manner of forms are obtained. Fourier has shown that *any* single-valued periodic curve may be built up by combining a number of simple sine-curves. Analytically this means that any single-valued periodic function may be expressed as the sum of a series of sine-terms; thus,

$$y = f(x) = A \sin \alpha x + B \sin 2 \alpha x + C \sin 3 \alpha x + \dots \text{etc.,}$$
$$+ P \cos \alpha x + Q \cos 2 \alpha x + R \cos 3 \alpha x + \dots \text{etc.,}$$

where f is a single-valued function. This is true for any single-valued periodic function, even one represented by an irregular series of straight lines. Each coefficient A, B, C, etc., is independent of x and has only one value which Fourier has shown how to find.

CHAPTER III.

CIRCUITS CONTAINING RESISTANCE AND SELF INDUCTION.

IN the first chapter the equation of energy for a circuit containing self-induction and resistance was derived, and from it the equation of electromotive forces

$$e = Ri + L\frac{di}{dt}; \tag{9}$$

that is, the electromotive force applied to the circuit is equal to the sum of the electromotive force necessary to overcome resistance and the electromotive force necessary to overcome the counter electromotive force of self-induction.

This equation of electromotive forces, being regarded as a differential equation containing three variables e, i, and t (of which the general type of the first order is

$$P\,dx + Q\,dy + S\,dz = 0, \tag{13}$$

where P, Q, and S are any functions of x, y, and z) does not satisfy the condition of integrability. That condition, which must hold true when there exists a single integral equation of which (13), or a multiple of (13), is the exact differential,* is

$$P\left(\frac{dQ}{dz}-\frac{dS}{dy}\right)+Q\left(\frac{dS}{dx}-\frac{dP}{dz}\right)+S\left(\frac{dP}{dy}-\frac{dQ}{dx}\right)=0.$$

If we put (9) in the form of (13), we have

$$0\,de - L\,di + (e - R\,i)\,dt = 0.$$

Here e, i, and t correspond to x, y, and z respectively, and $P=0$, $Q=-L$, $S=e-R\,i$.

The criterion of integrability reduces to

$$-L\,(1-0)=0, \quad \text{or} \quad -L=0,$$

and is not satisfied.

The meaning of this is that, unless some relation exists between two or more of the variables, there is no single equation of which (9) is the exact differential.

We know that the impressed E. M. F., e, has one single value at any particular point of time, and may therefore be expressed as a function of the time thus,

$$(14) \qquad e=f(t),$$

where f is any arbitrary single-valued function.

By equating (14) to (9) the equation (9) of E. M. F.'s is reduced to a linear equation, having constant coefficients with the second member equal to $\frac{f(t)}{L}$. Thus,

$$(15) \qquad \frac{di}{dt}+\frac{R}{L}i=\frac{1}{L}f(t).$$

---

* See Johnson's Differential Equations, p. 270.

The general type of this equation is

(16) $$\frac{dy}{dx} + Py = Q,$$

where P and Q may be functions of x only. The solution of equation (16), which is a linear differential equation of the first order,* is

(16a) $$y = \epsilon^{-\int P\,dx} \int \epsilon^{\int P\,dx}\, Q\,dx + c\,\epsilon^{-\int P\,dx}.$$

ϵ denotes the base of the Naperian system of logarithms and is equal to 2.718. c is the arbitrary constant of integration. Both of these letters will be thus used whenever they occur.

With the particular values of the coefficients in (15) its solution is, therefore,

(17) $$i = \frac{1}{L}\epsilon^{-\frac{Rt}{L}} \int \epsilon^{\frac{Rt}{L}} f(t)\,dt + c\,\epsilon^{-\frac{Rt}{L}}.$$

This is the general solution for the current flowing in a circuit containing resistance and self-induction and any impressed E. M. F.

The integration indicated in (17) can only be performed when we assume e to be some particular function of t. We proceed then to assume several ways in which the E. M. F. varies with the time.

Case I. Dying Away of Current on Removal of E. M. F. from a Circuit containing Resistance and Self-induction.

Suppose that a current has been flowing in a circuit until it has reached its steady state, and that the source of E. M. F. is then suddenly removed while the resistance

* See Johnson's Differential Equations, p. 31.

and self-induction remain the same. The equation of electromotive forces (9) becomes, under this hypothesis,

$$0 = Ri + \frac{L\,di}{dt}.$$

The solution of this equation is readily found since the variables admit of separation. Thus,

$$\frac{di}{i} = -\frac{R}{L}\,dt.$$

$$\therefore \log{}^{*} \frac{i}{c} = -\frac{Rt}{L},$$

or

$$i = c\,\epsilon^{-\frac{Rt}{L}}.$$

The constant of integration c is determined by the particular supposition introduced that when we begin to count the time, the current has its steady value I. This gives $c = I$. Hence we have

(18) $$i = I\epsilon^{-\frac{Rt}{L}}.$$

Referring to the general solution (17), we might have written (18) at once. For as $f(t) = 0$, [see (14),] the integral vanishes, and we have (18) as an immediate result.

This equation (18) is graphically represented in Fig. 9, where the ordinates represent the values of the current at any time after the E. M. F. is removed. The self-induction of the circuit prevents the current from falling immediately to zero. It is evident that it would do so if there were no self-induction from equation (18); for, if we make $L = 0$, i becomes zero. The current which flows after the removal of the E. M. F. is called the extra current of self-induction. The energy required to cause such a current to flow is that energy which was previously stored up in the field and is

* Naperian logarithm (base ϵ) is used here and in corresponding cases which follow.

now returned to the circuit. When t has the value $\frac{L}{R}$ the exponent of ϵ becomes minus unity, and we have the rela-

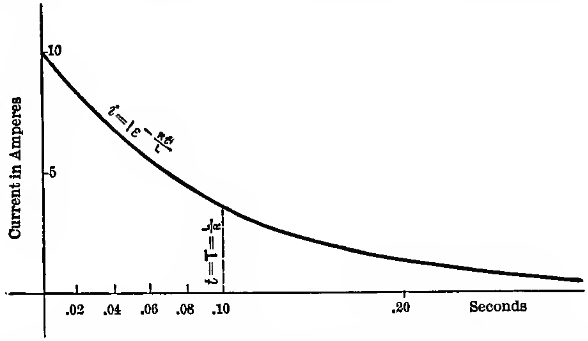

FIG. 9.—CURVE SHOWING THE DYING AWAY OF CURRENT AT ANY TIME AFTER THE REMOVAL OF THE IMPRESSED E. M. F. FROM A CIRCUIT WHOSE RESISTANCE R IS .1 OHM AND COEFFICIENT OF SELF-INDUCTION L IS .01 HENRY.

tion $\frac{I}{i} = \epsilon = 2.71828$. $\frac{L}{R}$ represents, therefore, the time that it takes for the current to fall to one ϵth part, that is to $\frac{1}{2.71828}$ of its original value. This is sometimes called the *time-constant* of the circuit, and denoted by T, that is $\frac{L}{R} = T$. The curve represents an exponential function of the time and approaches the x-axis as an asymptote. This means that the current becomes smaller and smaller, but is never zero until an infinite time has elapsed.

GEOMETRICAL METHOD OF CONSTRUCTING THE LOGARITHMIC CURVE.

The following method shown in Figs. 10 and 11 will be found to be a convenient way to construct a curve graphically whose equation is of the form

(19) $$y = c\,\epsilon^{-ax},$$

where c and a have any real values whatever.

Lay off $\overline{OA}$ equal to c. Then $\overline{OA}$ is the value of y when $x = 0$ and may be called y_0; that is, $y_0 = c = \overline{OA}$.

Lay off $\overline{OB}$ equal to $c\,\epsilon^{-ax_1}$. Then $\overline{OB}$ is the value of y when $x = x_1$, and may be called y_1; that is, $y_1 = c\,\epsilon^{-ax_1} = \overline{OB}$.

Hence $\dfrac{y_0}{y_1} = \dfrac{\overline{OA}}{\overline{OB}} = \epsilon^{ax_1}$.

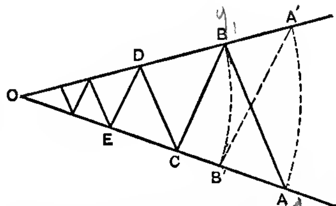

FIG. 10.—GRAPHICAL METHOD OF CONSTRUCTING A LOGARITHMIC CURVE.

If arcs AA' and BB' are described from the centre O, and a line $\overline{BC}$ drawn parallel with $\overline{A'B'}$, thence another line $\overline{CD}$ drawn parallel with $\overline{AB}$, and so on, lines parallel with $\overline{A'B'}$ and with $\overline{AB}$ being alternately drawn, as in the figure, then the distances $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, $\overline{OD}$, etc., will represent the values of y respectively as x takes the values $0, x_1, 2x_1, 3x_1$, etc. For if y_0, y_1, y_2, y_3, etc., denote the values of y when x takes the values $0, x_1, 2x_1, 3x_1$, respectively, we have

$$y_0 = c.$$

$$y_1 = c\,\epsilon^{-ax_1}. \quad \therefore \frac{y_0}{y_1} = \epsilon^{ax_1}.$$

$$y_2 = c\,\epsilon^{-2ax_1}. \quad \therefore \frac{y_1}{y_2} = \epsilon^{ax_1}.$$

$$y_3 = c\,\epsilon^{-3ax_1}. \quad \therefore \frac{y_2}{y_3} = \epsilon^{ax_1}.$$

Hence $$\frac{y_0}{y_1} = \frac{y_1}{y_2} = \frac{y_2}{y_3} = \frac{y_3}{y_4} = \text{etc.} = \epsilon^{ax_1}.$$

From the construction of the figure, and remembering that $\overline{OA} = y_0$ and $\overline{OB} = y_1$, we see

$$\frac{\overline{OA}}{\overline{OB}} = \frac{\overline{OB}}{\overline{OC}} = \frac{\overline{OC}}{\overline{OD}} = \frac{\overline{OD}}{\overline{OE}} = \text{etc.} = \epsilon^{ax_1}.$$

Hence $\qquad y_2 = \overline{OC}, \qquad y_3 = \overline{OD}$, etc.

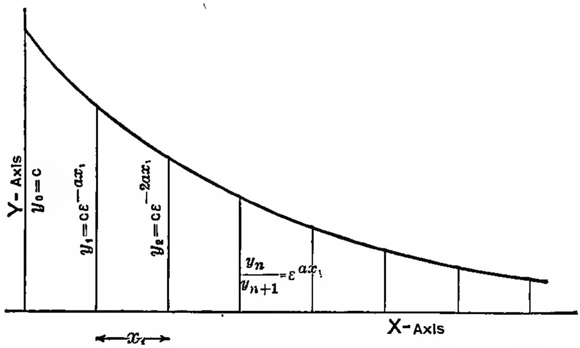

FIG. 11.—LOGARITHMIC CURVE.

Therefore to construct the curve $y = c\epsilon^{-ax}$, Fig. 11, we may proceed as follows: Upon two intersecting lines, as in Fig. 10, lay off the distances $y_0 = c$, and $y_1 = c\epsilon^{-ax_1}$, which latter must be calculated, and obtain the values of $\overline{OC}$, $\overline{OD}$, etc., as described. Then y_0, y_1, y_2, etc., or $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, etc., will be the successive ordinates of the logarithmic curve, Fig. 11, at distances 0, x_1, $2x_1$, $3x_1$, etc., and the curve may be drawn.

CASE II. ESTABLISHMENT OF A CURRENT ON INTRODUCTION OF A CONSTANT ELECTROMOTIVE FORCE INTO A CIRCUIT CONTAINING RESISTANCE AND SELF-INDUCTION.

Suppose a source of constant E. M. F. is suddenly introduced into a circuit of resistance R and self-induction L. The differential equation in this case is

$$E = Ri + L\frac{di}{dt}, \tag{20}$$

where E is a constant. The variables may be separated here as in the previous case, thus:

$$\frac{di}{i - \frac{E}{R}} = -\frac{R}{L}dt,$$

and $$\log\frac{1}{c}\left(i - \frac{E}{R}\right) = -\frac{Rt}{L}.$$

Therefore $$i = \frac{E}{R} + c\,\epsilon^{-\frac{Rt}{L}}.$$

The constant of integration, c, is determined by the condition that, when $t = 0$, $i = 0$, and therefore $c = -\frac{E}{R}$. We have then as a result,

(21) $$i = \frac{E}{R}\left(1 - \epsilon^{-\frac{Rt}{L}}\right) = I\left(1 - \epsilon^{-\frac{Rt}{L}}\right).$$

Referring to the general solution (17) we might have substituted $f(t) = E$, a constant, and written at once equation (21). For in this case we easily find the required integral:

$$\int E\,\epsilon^{\frac{Rt}{L}}dt = E\frac{L}{R}\epsilon^{\frac{Rt}{L}}.$$

Multiplying this by the coefficient $\frac{1}{L}\epsilon^{-\frac{Rt}{L}}$ (17) becomes

$$i = \frac{E}{R} + c\,\epsilon^{-\frac{Rt}{L}}.$$

Replacing c by $-\frac{E}{R}$, we have

$$i = \frac{E}{R}\left(1 - \epsilon^{-\frac{Rt}{L}}\right),$$

a result identical with (21).

Here we notice that, if the self-induction is zero, the equation becomes simply Ohm's law; that is, it is the self-induction of the circuit which prevents the current from reaching its full value immediately after the introduction of the E. M. F.

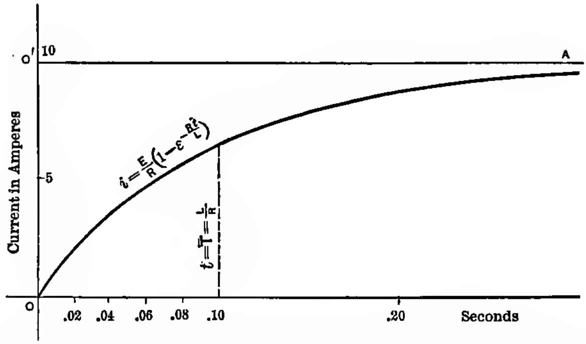

FIG. 12.—CURVE SHOWING THE ESTABLISHMENT OF CURRENT AT ANY TIME AFTER THE INTRODUCTION OF AN E. M. F. INTO A CIRCUIT WHOSE RESISTANCE R IS .1 OHM AND COEFFICIENT OF SELF-INDUCTION L IS .01 HENRY.

The increase of the current with the time is shown by the curve Fig. 12. This is a logarithmic curve similar to that in Fig. 9, with the ordinates measured downward from the horizontal line $O'A$, at a distance above the axis equal to the maximum value I, of the current.

CASE III. HARMONIC IMPRESSED E. M. F. IN A CIRCUIT CONTAINING A RESISTANCE AND SELF-INDUCTION.

Let us now suppose that in a circuit containing resistance and self-induction there is a simple harmonic impressed E. M. F., that is that the E. M. F. is a sine-function of the time, thus:

$$(22) \qquad e = f(t) = E \sin \omega t.$$

Here E is the amplitude or maximum value of the impressed E. M. F., and ω is the angular velocity, equivalent

to $2\pi n$, or $\frac{2\pi}{T}$, where n denotes the number of complete periods per second, and T the time of one complete period.

The general solution for the current, equation (17), is

$$(17) \qquad i = \frac{1}{L} \epsilon^{-\frac{Rt}{L}} \int \epsilon^{\frac{Rt}{L}} f(t)\, dt + c\, \epsilon^{-\frac{Rt}{L}}.$$

Substituting in (17) the value for $f(t)$ in (22), the general expression becomes, according to the particular hypothesis of a sine E. M. F.,

$$(23) \qquad i = \frac{E}{L} \epsilon^{-\frac{Rt}{L}} \int \epsilon^{\frac{Rt}{L}} \sin \omega t\, dt + c\, \epsilon^{-\frac{Rt}{L}}.$$

Before integrating this equation we will first obtain the general integrals

$$\int \epsilon^{ax} \sin (\beta x + \theta) dx \quad \text{and} \quad \int \epsilon^{ax} \cos (\beta x + \theta)\, dx.$$

Applying the formula for integrating by parts,

$$\int u\, dv = u\, v - \int v\, du,$$

these integrals become

$$\int \sin (\beta x + \theta) \,.\, \epsilon^{ax}\, dx$$
$$= \sin (\beta x + \theta) \,.\, \frac{\epsilon^{ax}}{\alpha} - \frac{\beta}{\alpha} \int \epsilon^{ax} \cos (\beta x + \theta)\, dx,$$

$$\int \cos (\beta x + \theta) \,.\, \epsilon^{ax}\, dx$$
$$= \cos (\beta x + \theta) \,.\, \frac{\epsilon^{ax}}{\alpha} + \frac{\beta}{\alpha} \int \epsilon^{ax} \sin (\beta x + \theta)\, dx.$$

Eliminating $\int \epsilon^{ax} \cos (\beta x + \theta)\, dx$ between these two equations, we obtain as one of the integrals sought

$$(24) \qquad \int \epsilon^{ax} \sin (\beta x + \theta)\, dx$$
$$= \frac{\epsilon^{ax}}{\alpha^2 + \beta^2} \left\{ \alpha \sin (\beta x + \theta) - \beta \cos (\beta x + \theta) \right\}.$$

Eliminating $\int \epsilon^{ax} \sin(\beta x + \theta)\, dx$ between the same two equations, we obtain in the same way the integral

$$(25) \qquad \int \epsilon^{ax} \cos(\beta x + \theta)\, dx$$

$$= \frac{\epsilon^{ax}}{\alpha^2 + \beta^2} \left\{ \alpha \cos(\beta x + \theta) + \beta \sin(\beta x + \theta) \right\}.$$

Replacing α by $\frac{R}{L}$, β by ω, θ by 0, and x by t, in equation (24), we have the integration indicated in (23), and equation (23) then becomes

$$(26) \qquad i = \frac{E}{L\left(\frac{R^2}{L^2} + \omega^2\right)} \left(\frac{R}{L} \sin \omega t - \omega \cos \omega t\right) + c\,\epsilon^{-\frac{Rt}{L}}.$$

This may be written in simpler form by the use of the trigonometric formula

$$(27) \qquad A \sin\theta + B \cos\theta = \sqrt{A^2 + B^2} \sin\left(\theta + \tan^{-1}\frac{B}{A}\right).$$

This formula is established as follows:

$$A \sin\theta + B\cos\theta$$

$$= \sqrt{A^2 + B^2}\left(\frac{A}{\sqrt{A^2 + B^2}} \sin\theta + \frac{B}{\sqrt{A^2 + B^2}} \cos\theta\right).$$

If $\tan\phi = \frac{B}{A}$, then $\sin\phi = \frac{B}{\sqrt{A^2 + B^2}}$, and $\cos\phi = \frac{A}{\sqrt{A^2 + B^2}}$.

Making these substitutions, we have

$$A \sin\theta + B\cos\theta = \sqrt{A^2 + B^2}\,(\cos\phi \sin\theta + \sin\phi\cos\theta)$$
$$= \sqrt{A^2 + B^2}\,\sin(\theta + \phi),$$

which establishes the truth of (27).

Reducing equation (26) to its simplest form by means of formula (27), we have from (26) the value of the current at any instant of time.

$$(28) \quad i = \frac{E}{\sqrt{R^2 + L^2 \omega^2}} \sin\left(\omega t - \tan^{-1}\frac{L\omega}{R}\right) + c\,\epsilon^{-\frac{Rt}{L}}.$$

DISCUSSION OF THE CURRENT EQUATION.

After a very short time the exponential term in this equation, containing the arbitrary constant of integration, becomes inappreciably small, and may be neglected. Just what effect the exponential term has during this short time will be considered later. The equation shows that, where there is an impressed sine electromotive force in a circuit, the current is likewise a sine-function of the time, and that the current lags behind the electromotive force by an angle whose tangent is $\frac{L\omega}{R}$. If there is no self-induction and $L = 0$, equation (28) becomes

$$i = \frac{E}{R} \sin \omega t,$$

which is a direct result of Ohm's law. Thus the self-induction not only causes the current to lag behind the impressed E. M. F., but also diminishes the maximum value of the current.

When $\sin\left(\omega t - \tan^{-1}\frac{L\omega}{R}\right)$ becomes unity, the current has its maximum value I, and

$$(29) \qquad I = \frac{E}{\sqrt{R^2 + L^2 \omega^2}}.$$

The term "impedance" has been applied to the expression $\sqrt{R^2 + L^2 \omega^2}$, the apparent resistance of a circuit containing ohmic resistance and self-induction, and an impressed sine electromotive force.

The equation (29) may be written

$$(30)\qquad \text{Maximum current} = \frac{\text{Maximum E. M. F.}}{\text{Impedance}}.$$

Since virtual current $= \frac{1}{\sqrt{2}}$ maximum current, and virtual E. M. F. $= \frac{1}{\sqrt{2}}$ maximum E. M. F., see equation (12), we may write

$$(31)\qquad \text{Virtual current} = \frac{\text{Virtual E. M. F.}}{\text{Impedance}}.$$

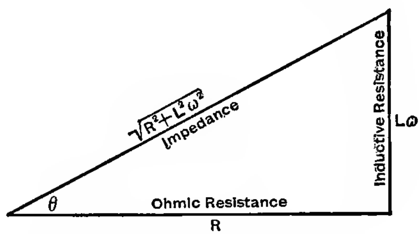

FIG. 13.—VALUE OF IMPEDANCE.

The value of impedance is graphically represented in Fig. 13. $L\,\omega$ is sometimes called the *inductive resistance* in contradistinction to the *ohmic resistance* R. (See note p. 59.)

It has been shown above that the tangent of the angle of lag is $\frac{L\,\omega}{R}$. The angle of lag is therefore represented by θ in Fig. 13.

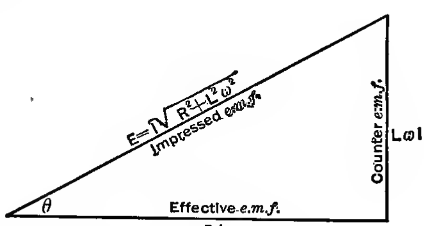

FIG. 14.—VALUE OF IMPRESSED E. M. F.

The triangle may be drawn so that the three sides represent E. M. F. as in Fig. 14. Here $R\,I$ represents the

E. M. F. necessary to overcome the ohmic resistance, and is in the same direction as the current. $L \omega I$ is at right angles to this and represents the counter E. M. F. of self-induction. $I \sqrt{R^2 + L^2 \omega^2}$ is the impressed electromotive force E. θ is the angle by which the current lags behind the impressed E. M. F.

Full discussion of the triangles of current and E. M. F. is given in the graphical treatment of circuits with resistance and self-induction, Chap. XV.

It is convenient to consider the impedance as a resistance, and the propriety of doing so is shown by its dimensions, which are the same as those of resistance, that is a velocity in the electromagnetic system of units.

The dimensions of resistance, R, are $\dfrac{length}{time} = velocity$.

The dimension of the coefficient of self-induction, L, is *length*. The dimension of an angular velocity ω is $\dfrac{1}{time}$.

Therefore the dimensions of $L \omega$ are $\dfrac{length}{time} = velocity$, and thus the impedance has the same dimensions as a resistance.

EXPLANATION OF THE EXPONENTIAL TERM.

Let us return to the solution for current, equation (28), and consider the effect of the exponential term, $c \epsilon^{-\frac{Rt}{L}}$, during the short time after "make," that is, after the introduction into the circuit of a simple harmonic impressed electromotive force. The equation (28) for current may be written

$$(32) \qquad i = I \sin \psi + c \epsilon^{-\frac{Rt}{L}};$$

where $$I = \frac{E}{\sqrt{R^2 + L^2 \omega^2}},$$

and $$\psi = \omega t - \tan^{-1} \frac{L \omega}{R};$$

that is, I represents the maximum value and ψ the phase of the current. The E. M. F. is introduced at a time t_1. At that time the current is zero, for the circuit is just made. If we call ψ_1 the value of ψ when $t = t_1$, at the introduction of the E. M. F. equation (32) becomes

$$0 = I \sin \psi_1 + c\, \epsilon^{-\frac{Rt_1}{L}},$$

(33) and $$c = -I \epsilon^{+\frac{Rt_1}{L}} \sin \psi_1.$$

Substituting this value of c in (32), the equation for current becomes

(34) $$i = I \sin \psi - I \epsilon^{-\frac{R}{L}(t-t_1)} \sin \psi_1.$$

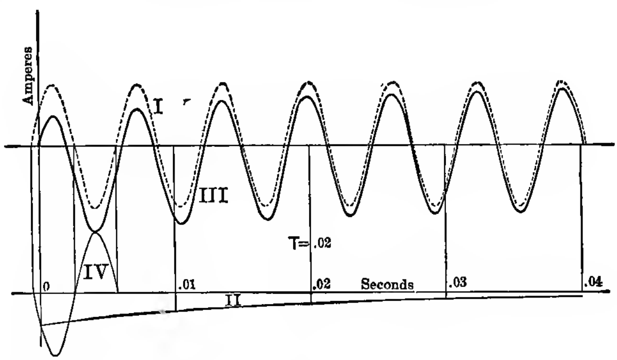

FIG. 15.—CURVE SHOWING THE EFFECT OF THE EXPONENTIAL TERM $c\,\epsilon^{-\frac{Rt}{L}}$ UPON THE CURRENT AT THE MAKE, IN A CIRCUIT WHERE $L = 1$ HENRY, $R = 50$ OHMS, $\omega = 1000$, $\psi_1 = 30°$.

This equation may best be explained by referring to Fig. 15, which represents the plot of the equation. The particular values assumed in this case are $L = 1$ henry, $R = 50$ ohms, $\omega = 1000$, and $\psi_1 = 30°$. The resultant current curve III. is made up of two component parts, $I \sin \psi$, and $-I \epsilon^{-\frac{R}{L}(t-t_1)} \sin \psi_1$, which are represented by the curves

I. and II. respectively. Curve I. is a sine-curve and curve II. a logarithmic curve, the effect of which upon the resultant current becomes inappreciable after a very short space of time, in this particular case after five or ten periods. The initial value of this logarithmic curve is equal and opposite to the value of the ordinate of the component sine-curve I. at the time t, when the E. M. F. is introduced. This is evident from the equation, since the initial value of the logarithmic curve is $-I \sin \psi_1$, and the value of the sine-curve, when $t = t_1$, is $+ I \sin \psi_1$.

If another curve IV. is constructed so that its ordinates represent the initial values of the logarithmic curve, when the E. M. F. is introduced at different points in the period, it is seen to be simply a sine-curve, corresponding with the component curve I. but reversed, or, what is the same thing, differing from it by 180° in phase.

To conclude, we see that the effect of the exponential term in the equation is a maximum if the E. M. F. is introduced at that point of its phase at which the current has its maximum value when everything has reached its permanent state; this term has no effect if the E. M. F. is introduced at that point of its phase at which the current has its zero value when everything has reached its permanent state.

Case IV.—Periodic E. M. F. which is not Harmonic, in Circuits containing Resistance and Self-induction.

In Case III. the solution was given for a circuit containing an impressed E. M. F. which was a simple sine-function of the time. Now let us suppose that the E. M. F. does not follow a simple sine law, but that it is the sum of two components each following a sine law, that is,

$$e = E_1 \sin \omega t + E_2 \sin (b \, \omega t + \theta). \tag{35}$$

Substituting in the general expression for current (17) this value for $f(t)$, we have

$$(36) \qquad i = \frac{E_1}{L}\epsilon^{-\frac{Rt}{L}}\int \epsilon^{\frac{Rt}{L}} \sin \omega t \, dt$$

$$+ \frac{E_2}{L}\epsilon^{-\frac{Rt}{L}}\int \epsilon^{\frac{Rt}{L}} \sin (b\,\omega t + \theta)\, dt + c\,\epsilon^{-\frac{Rt}{L}}.$$

Performing the indicated integrations by use of the formula of integration (24), we have

$$(37) \qquad i = \frac{E_1}{L\left(\frac{R^2}{L^2} + \omega^2\right)} \left\{ \frac{R}{L} \sin \omega t - \omega \cos \omega t \right\}$$

$$+ \frac{E_2}{L\left(\frac{R^2}{L^2} + b^2\omega^2\right)} \left\{ \frac{R}{L} \sin (b\,\omega t + \theta) - b\,\omega \cos (b\,\omega t + \theta) \right\}$$

$$+ c\,\epsilon^{-\frac{Rt}{L}}.$$

Simplifying by formula (27) this becomes

$$(38) \qquad i = \frac{E_1}{\sqrt{R^2 + L^2\omega^2}} \sin \left(\omega t - \tan^{-1}\frac{L\omega}{R}\right)$$

$$+ \frac{E_2}{\sqrt{R^2 + L^2 b^2 \omega^2}} \sin \left(b\,\omega t + \theta - \tan^{-1}\frac{L b\omega}{R}\right)$$

$$+ c\,\epsilon^{-\frac{Rt}{L}}.$$

By this equation it is seen that each simple sine impressed E. M. F. gives rise to a simple sine term in the resulting current equation. The result may therefore easily be extended, and we may say that, if there are n simple sine impressed E. M. F.'s of the form $E \sin (b\,\omega t + \theta)$, where

E, b, and θ have different values in each component term, the current equation will be the sum of n terms of the form

$$\frac{E}{\sqrt{R^2 + L^2 b^2 \omega^2}} \sin \left\{ b\,\omega\,t + \theta - \tan^{-1} \frac{L\,b\,\omega}{R} \right\}$$

plus the term $c\,\epsilon^{-\frac{Rt}{L}}$ containing the arbitrary constant.

Here E, b, and θ have the same values in each term as they do in the corresponding term of the impressed E. M. F.

Expressing the current by a summation, we have

$$(39)\; i = \sum_{E, b, \theta.} \frac{E}{\sqrt{R^2 + L^2 b^2 \omega^2}} \sin \left\{ b\,\omega\,t + \theta - \tan^{-1} \frac{L\,b\,\omega}{R} \right\} + c\,\epsilon^{-\frac{Rt}{L}}$$

when the impressed E. M. F. is

$$(40) \quad e = \sum_{E, b, \theta.} E \sin\,[b\,\omega\,t + \theta].$$

In these sums E, b, and θ may have n values, but they must be the same values in each sum, giving rise to the same number of terms in each.

It was first shown by Fourier that such a sum of simple sine terms as that represented in equation (39) may express any single-valued function whatever, and thus we see that the equation expresses the most general case of a current flowing in a circuit with resistance and self-induction, and may represent the current caused by any E. M. F. whatsoever.

The consideration of this most general expression for the current will be deferred until the case has been taken up where the circuit not only contains resistance and self-induction, but also a condenser.

NOTE.—Since the first publication of this volume the quantity $L\omega$ has been termed *reactance*, and $L\,\omega\,I$ the *reactive electromotive force*. The component electromotive force in phase with the current may be termed the *power electromotive force*. The ohmic electromotive force RI is a power electromotive force. [*Effective* now usually means *virtual*, see p. 38.]

CHAPTER IV.

INTRODUCTORY TO CIRCUITS CONTAINING RESISTANCE AND CAPACITY.

CONTENTS:—Plan to be followed. Charge. Law of force. Unit charge. Work in moving a charge. Potential. Capacity. Energy of charge. Condenser,—energy of and capacity of. Capacity of parallel plates; of continuous conductor. Equation of energy, in terms of i; in terms of q. Equation of E. M. F.'s.

IN the first chapter the fundamental principles necessary to lead up to the derivation of the equation of energy for circuits containing resistance and self-induction only were given; then followed, in the third chapter, the solution of this differential equation, which enabled us to ascertain the current flowing in the circuit at any time. Following a similar plan, there will be given in this chapter the necessary fundamental principles which lead up to the derivation of the differential equation of energy for circuits containing resistance and capacity, and in the following chapter the general solution of this differential equation and its application to various particular cases.

LAW OF FORCE.

Every one is familiar with the fact that bodies may be charged with electricity, and that two like charges repel and two unlike charges attract one another. It was found from experiment by Coulomb that if we have two charges, each concentrated at a point, the force of attraction or

repulsion between them varies directly with the product of the two charges and inversely as the square of the distance between the two points, that is,

$$F \propto \frac{q\,q'}{r^2}.$$

where q and q' represent the quantities of the charges, r the distance, and F the force between them. When the quantities considered have the same sign, the product $q\,q'$ is positive, and therefore a force of repulsion has a positive sign. Similarly a force of attraction has a negative sign.

If the distance between these points is unity, the charges being equal, and if the force between them is a unit force, each charge is called a *unit charge.* The definition of the electrostatic unit of quantity of electricity, in the C. G. S. system, is then: that quantity which, when placed at a distance of one centimeter from an equal quantity (in a medium whose specific inductive capacity is unity —that is, in air or vacuo), repels it with the force of one dyne. Where these units are used, and the medium is a vacuum, the law of force may be written

$$F = \frac{q\,q'}{r^2}.$$

Where the medium is not a vacuum, the force is found to be less and equal to

$$F = \frac{q\,q'}{\kappa\,r^2},$$

where κ is a constant quantity called the *specific inductive capacity* of the medium.

POTENTIAL.

Since there exists a force between two charges of electricity, mechanical work is done if either is moved so as to change the distance between them. The work done in

moving any body against a uniform force is equal to the product of the force and the distance through which the body is moved *against that force*. The force between the electrical charges q and q' is $\frac{q\,q'}{r^2}$. If they be moved in any direction whatsoever, so that the distance r between them is changed to $r + dr$, the work done in moving them is the product of the force $\frac{q\,q'}{r^2}$ and the change in the distance dr, since the force may be considered constant throughout the small distance dr. Therefore the work is

$$dW = \frac{q\,q'}{r^2}\,dr.$$

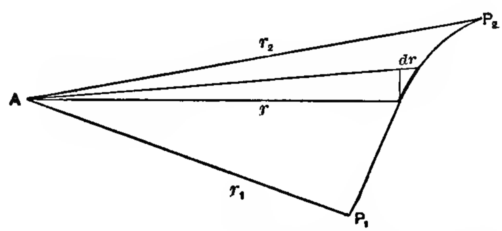

FIG. 16.—WORK DONE IN MOVING A CHARGED BODY.

Suppose a charge q is situated at the point A (Fig. 16), and a charge q' is moved from the point P_1 to P_2. The work done by the electric force in moving the charge is

$$W = \int_{r_1}^{r_2} \frac{q\,q'}{r^2}\,dr = qq'\left(\frac{1}{r_1} - \frac{1}{r_2}\right),$$

or, the work done against the electric force is $qq'\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$.

It is seen that the work done in moving a charge from one point to another is independent of the path by which it is moved, and simply depends on the initial and final distances between the charge q and q'. If the distance r_1 is infinite (meaning that the charge q' is carried from an in-

finite distance to a point at a distance r_2), the work done against the electric force becomes simply

$$W = \frac{q\,q'}{r_2}.$$

If q' is unity and a unit charge is moved, the work becomes

$$W = \frac{q}{r_2}.$$

It is seen that each point in the region surrounding an electric charge possesses a certain characteristic which determines the amount of work done in bringing a charge from infinity to that point. This characteristic of the point has been called its *potential.* The potential V at a point is therefore defined as the work done in moving a unit positive charge from an infinite distance to that point; thus, $V = \frac{q}{r}$. This potential is positive when the work is positive, that is, when work is done, in moving the charge, by some agent external to the system.

The potential at a point due to a number of charges, each concentrated at a point, is the sum of the potentials at that point due to each charge independently; thus,

$$V = \sum \frac{q}{r}.$$

If there is a charge distributed upon any surface and dq is the charge upon an element of that surface, the potential at any point due to this charged surface is equal to the sum of the potentials due to each elemental charge; that is,

$$V = \int \frac{dq}{r}.$$

The potential at every point of a good conductor is the same, since the electricity will so distribute itself on the body that no work would be done by transferring a

charge from one point of the conductor to another point of it. This potential V is called *the potential at the conductor*, and the conductor is said to be *at potential V*. The potential at a conductor may be due partly or wholly to the charge on the conductor itself.

CAPACITY OF A CONDUCTOR.

The potential of a charged body is directly proportional to its charge, that is, $V \propto q$, or $q = C\,V$, where C is some constant; for, suppose the body possesses a unit charge and its potential is V; a second unit charge brought from infinity to the body doubles its original charge. The potential is then $2\,V$, for the potential is the work done in bringing a unit charge from infinity to the point, and the work in bringing a unit charge to a body with a quantity $2q$ is twice the work in bringing a unit charge to a body with a quantity q. We thus see that q is proportional to V, and is consequently equal to V multiplied by some constant, that is,

$$q = CV. \tag{41}$$

If a body is charged to a unit potential and the quantity is q,

$$q = C.$$

C is therefore defined as the quantity of electricity upon a body when at a unit potential. This is called the *capacity* of the conductor. The capacity depends upon the size and geometrical form of the conductor and the specific inductive capacity of the surrounding medium.

ENERGY OF A CHARGED CONDUCTOR.

Suppose a body is charged with a quantity of electricity q, and is at a potential V. The work done in bringing a unit quantity of electricity from an infinite distance up to the body is V by definition. (This is provided q is so large

in comparison with a unit quantity that its potential is not appreciably altered by the addition of the unit quantity.) If, under the same conditions, we bring up, not a unit quantity, but a quantity dq, the work done is Vdq, and this represents the increment of the energy of the charge q. That is,

$$dW = V dq. \tag{42}$$

Referring to equation (41), we may always replace V by its equal $\frac{q}{C}$, or dq by its equal $C\,dV$, and obtain the equations

$$dW = \frac{q\,dq}{C},$$

and

$$dW = C\,V dV.$$

The integrals of these equations, taken between the limits zero and q, and zero and V, respectively, are

$$W = \frac{1}{2}\frac{q^2}{C},$$

$$W = \tfrac{1}{2}\,C\,V^2.$$

Since $q = C\,V$, each of these equations may be written

$$W = \tfrac{1}{2}\,q\,V. \tag{43}$$

Here W is the potential energy possessed by the charged body, as the limits of integration were taken from zero charge to charge q, and from zero potential to potential V.

CAPACITY AND ENERGY OF A CONDENSER.

A condenser is a device for increasing the capacity of a conductor by bringing it near another similar conductor, which is separated from it by any non-conducting medium or dielectric. This dielectric will be considered to be a

perfect non-conductor; that is, the condenser is not *leaky*. A condenser usually consists of two sets of parallel plates alternately connected, and separated by a distance very small as compared with the dimensions of the plates. The two sets of plates are usually called simply the two plates of the condenser. When the condenser is charged, the two plates have equal quantities of electricity upon them, but of the opposite sign.

The total energy of a charged condenser may readily be found by taking the algebraic sum of the energies of the charge on each plate, as given by the equation (43).

If the plates of a condenser have charges $+q$ and $-q$ at potentials V_1 and V_2, respectively, the total energy is

$$W = \tfrac{1}{2} q V_1 - \tfrac{1}{2} q V_2 = \tfrac{1}{2} q (V_1 - V_2); \tag{44}$$

that is, the energy of a charged condenser is equal to one-half the product of the charge of one of the plates and the difference of potential between the plates. If this difference of potential between the plates is simply V, the expression for the energy of a charged condenser is

$$W = \tfrac{1}{2} q V. \tag{45}$$

The capacity C of a condenser is the quantity of electricity on one plate when there is a unit difference of potential between the plates; and when there is a difference of potential V the charge is

$$q = C V. \tag{46}$$

It can be shown that the capacity of a condenser, composed of parallel plates of equal area, whose distance apart is small as compared with the dimensions of the plates, is directly proportional to the area of the plates, and inversely

proportional to the distance between them, and that the capacity is

(47) $$C = \frac{A}{4\,\pi\,d},$$ [See note, p. 69.]

where A is the area of each plate and d the distance between the plates.

As the plates of a condenser approach nearer and nearer together, the capacity C becomes larger and larger. In the limit, when the plates come into contact, the capacity becomes infinite, which means that, no matter how much one plate is charged, there can exist no difference of potential between them. If, then, a circuit is a continuous conductor and has no condenser in it, it may be said to have a condenser of infinite capacity in series with it.

By combining equations (45) and (46) the energy of the charge of the condenser may be expressed in terms of the capacity and the potential V, or in terms of the capacity and the charge q. Thus,

(48) $$W = \tfrac{1}{2}\,C\,V^2 = \frac{1}{2}\frac{q^2}{C}.$$

The increment of the energy $d\,W$, as the potential and charge vary simultaneously, is

(49) $$d\,W = C\,V d\,V = \frac{q\,dq}{C}.$$

THE EQUATION OF ENERGY.

We can now write the equation of energy for an electric circuit having a resistance R, and having in series with that resistance a condenser of capacity C.

The total energy given to the circuit by the source of E. M. F. is $e\,i\,dt$; and that part of the energy used in heating the conductor in the time dt is $R\,i^2\,dt$, as shown in equations (5) and (4). The amount of energy required in the time dt to change the charge of the condenser is $\frac{d\,W}{dt}\,dt$.

Since, under the conditions supposed, these two are the only ways in which the energy imparted by the source is used, we have the equation of energy,

$$e\,i\,dt = R\,i^2\,dt + \frac{dW}{dt}\,dt. \tag{50}$$

We have seen that $dW = \frac{q\,dq}{C}$ (equation 49); therefore,

$$e\,i\,dt = R\,i^2\,dt + \frac{q}{C}\frac{dq}{dt}\,dt. \tag{51}$$

When a current i flows into a condenser for a time dt, the quantity which flows during this time is $i\,dt$, but this is the increment dq of the charge of the condenser, that is,

$$dq = i\,dt;$$

hence

$$q = \int i\,dt. \tag{52}$$

Substituting these values of q and i in equation (51) we may write the equation of energy in two forms, in terms of i or in terms of q, thus:

$$e\,i\,dt = R\,i^2\,dt + \frac{i\,dt\int i\,dt}{C}; \tag{53}$$

$$e\frac{dq}{dt}dt = R\left(\frac{dq}{dt}\right)^2 dt + \frac{q}{C}\frac{dq}{dt}\,dt. \tag{54}$$

Dividing (53) through by $i\,dt$ and (54) by its equal dq, we have

$$e = Ri + \frac{\int i\,dt}{C}; \tag{55}$$

$$e = R\frac{dq}{dt} + \frac{q}{C}. \tag{56}$$

These are equations of electromotive forces, where e is the impressed E. M. F. of the source, Ri or $R\frac{dq}{dt}$ the E. M. F. necessary to overcome the ohmic resistance, and $\frac{\int i\,dt}{C} = \frac{q}{C} = V$, the E. M. F. necessary to oppose the E. M. F. of the condenser.

When C is infinite, that is, as explained above, when the plates of the condenser come into contact, we have a circuit with resistance only, in which case equation (55) gives

$$e = Ri,$$

which is Ohm's law.

NOTE.—This expression for capacity, equation (47) (page 67), is true for C. G. S. electrostatic units. To find the value in electromagnetic or practical units, consult Appendix A, page 312.

CHAPTER V.

CIRCUITS CONTAINING RESISTANCE AND CAPACITY.

IN the previous chapter the equation of energy for a circuit containing ohmic resistance and capacity was derived, and, by dividing the equation of energy through by $i\,dt$ or dq, it was found that the equation of electromotive forces thus obtained may be expressed in terms of current, i, or charge, q, thus:

(55) $$e = Ri + \int \frac{i\,dt}{C}.$$

(56) $$e = R\frac{dq}{dt} + \frac{q}{C}.$$

Differentiating (55), to free it from the integral sign, and transposing, the two equations may be written:

(57) $$C\,de - R\,C\,di - i\,dt = 0.$$

(58) $$0\,de - R\,C\,dq + (e\,C - q)\,dt = 0.$$

Each of these equations is a differential equation of the first order with three variables, e, i, and t, and e, q, and t, respectively, of the form

$$P\,dx + Q\,dy + S\,dz = 0.$$

If there exists a single integral equation of which this is the exact differential, the condition of integrability *

$$P\left(\frac{dQ}{dz} - \frac{dS}{dy}\right) + Q\left(\frac{dS}{dx} - \frac{dP}{dz}\right) + S\left(\frac{dP}{dy} - \frac{dQ}{dx}\right) = 0$$

must be satisfied.

Applying this criterion of integrability to the equations

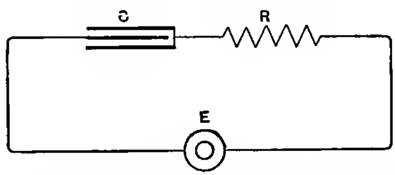

FIG. 17.—CIRCUIT HAVING OHMIC RESISTANCE AND CAPACITY.

(57) and (58), it is found that the condition is not satisfied by either equation. No single equation exists, therefore, of which (57) or (58) is an exact differential.

But, as was previously stated, we know that the electromotive force e may always be expressed as a single-valued function of the time, since it must have some one value at each point of time, and we have

(59) $$e = f(t),$$

where f is an arbitrary single-valued function. By differentiation (59) becomes

(60) $$\frac{de}{dt} = f'(t).$$

* See Johnson's Differential Equations, p. 270.

Equations (57) and (58) may now be written in the linear form thus:

$$\frac{di}{dt}+\frac{i}{R\,C}=\frac{1}{R}f'(t). \tag{61}$$

$$\frac{dq}{dt}+\frac{q}{R\,C}=\frac{1}{R}f(t). \tag{62}$$

The solutions of these linear equations * are

$$i=\frac{\epsilon^{-\frac{t}{R\,C}}}{R}\int\epsilon^{+\frac{t}{R\,C}}f'(t)\,dt+c_1\,\epsilon^{-\frac{t}{R\,C}}. \tag{63}$$

$$q=\frac{\epsilon^{-\frac{t}{R\,C}}}{R}\int\epsilon^{+\frac{t}{R\,C}}f(t)\,dt+c_2\,\epsilon^{-\frac{t}{R\,C}}. \tag{64}$$

The integrals here expressed cannot be found unless we know in what particular way the electromotive force varies with the time. When we know this, these equations will give the values of the current and charge at any time, provided the integral sought can be obtained. We will now assume several ways in which the E. M. F. varies with the time, which will allow the integration to be easily performed.

Case I. Discharge of a Condenser.

Suppose that a constant source of E. M. F., E, has been acting upon a circuit containing in series a resistance, and a condenser with capacity C, until everything has reached its steady state. No current will be flowing, and the condenser will be charged with a quantity Q, and have a difference of potential E at its terminals. Now suddenly remove the source of E. M. F. from the circuit and suppose its resistance then is R. The condenser will immediately

* See Johnston's Differential Equations, p. 31.

begin to discharge through the conductor, and we wish to find the value of the charge q and current i at any time after the discharge begins.

When the E. M. F. was removed from the circuit the impressed E. M. F., $e = f(t)$, became equal to zero at every point of time after the removal; hence, substituting

$$e = f(t) = 0 \tag{65}$$

in the general equations (63) and (64), we find that the integral vanishes, and we have the immediate results,

$$i = c_1 \epsilon^{-\frac{t}{RC}}.$$

$$q = c_2 \epsilon^{-\frac{t}{RC}}.$$

The arbitrary constants c_1 and c_2 are determined by the initial conditions. If the charge is Q when the time is zero, the charge equation becomes

$$q = Q \epsilon^{-\frac{t}{RC}}; \tag{66}$$

and since $dq = i\,dt$, the current equation becomes

$$i = -\frac{Q}{RC} \epsilon^{-\frac{t}{RC}}. \tag{67}$$

If, instead of substituting $e = f(t) = 0$ in the general solutions (63) and (64), we had substituted in the differential equations (61) and (62), it is seen that the second member of each becomes zero, and that the solutions are merely the "complementary functions," namely, the terms in the general solutions containing the arbitrary constants, as pointed out.

It may be of interest to derive the solution directly from the differential equations, since the variables easily admit of separation.

Equation (62), when $f(t) = 0$, is

$$\frac{dq}{dt} + \frac{q}{RC} = 0.$$

$$\text{Hence} \quad \frac{dq}{q} = -\frac{dt}{RC},$$

$$\text{and} \quad \log\frac{q}{c} = -\frac{t}{RC},$$

$$\text{or} \quad q = c\epsilon^{-\frac{t}{RC}},$$

which is identical with (67).

In Fig. 18 is shown a curve of discharge of a condenser for a particular case. The rapidity of discharge is shown by the value of the time-constant T, which gives the time in which the charge of the condenser is reduced to one ϵth of its initial value.

$$T = RC = 100 \times 10^9 \times 4 \times 10^{-15} = .0004 \text{ seconds.}$$

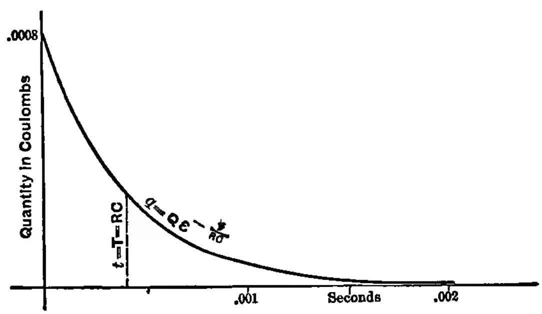

Fig. 18.—Curve showing Discharge of a Condenser whose Capacity $C = 4$ Microfarads, through a Resistance $R = 100$ Ohms.

Case II. Charge of a Condenser.

Suppose that a constant source of E. M. F., E, is suddenly introduced into a circuit, and that the resistance when it is introduced is R, the capacity of the condenser in series with the resistance being C. The values of the

current i and charge q at any time after the introduction of the E. M. F. will be given by equations (63) and (64) if we suppose

$$e = f(t) = E, \text{ a constant,} \tag{68}$$

and consequently $\dfrac{de}{dt} = f'(t) = 0.$

Substituting these values, (63) and (64) become

$$i = c_1 \epsilon^{-\frac{t}{RC}}. \tag{69}$$

$$q = CE + c_2 \epsilon^{-\frac{t}{RC}}. \tag{70}$$

Determining the constants of integration c_1 and c_2 by the condition that there was no charge in the condenser when $t = 0$, we have

$$c_2 = -CE.$$

But since $CE = Q$, the final charge of the condenser when everything has reached its steady state, (70) becomes

$$q = Q\left(1 - \epsilon^{-\frac{t}{RC}}\right), \tag{71}$$

and by the relation $dq = i\,dt$ equation (69) becomes

$$i = \frac{Q}{RC}\epsilon^{-\frac{t}{RC}}. \tag{72}$$

It is noticeable that the equations for the current (67) and (72) are identical in the case of charge and discharge of a condenser, except that the sign of i, i.e., the direction of the current, is reversed.

Equation (71) may easily be derived from the differential equation (62) directly, upon substituting $f(t) = E$, as the variables easily admit of separation; thus,

$$\frac{dq}{dt} + \frac{q}{RC} = \frac{E}{R}$$

may be written
$$\frac{dq}{q - CE} = -\frac{dt}{RC},$$

and
$$\log \frac{(q - CE)}{c_2} = -\frac{t}{RC}.$$

Hence
$$q = CE + c_2\,\epsilon^{-\frac{t}{RC}},$$

which is identical with (70).

The curve representing the charge of a condenser is shown in Fig. 19. The time-constant $RC = .0004$. The final charge is

$$Q = CV = 4 \times 10^{-16} \times 200 \times 10^{8} \times 10 = .0008 \text{ coulombs}.$$

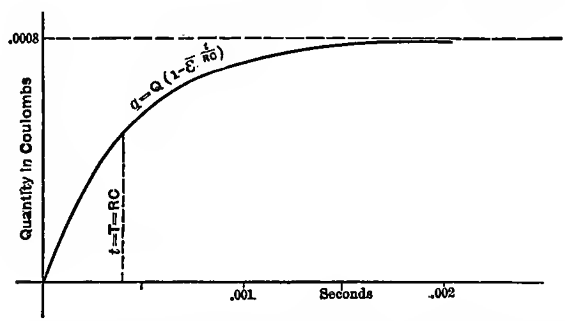

FIG. 19.—CURVE SHOWING THE CHARGE OF A CONDENSER WHOSE CAPACITY $C = 4$ MICROFARADS WHEN SUBJECTED TO A DIFFERENCE OF POTENTIAL OF 200 VOLTS THROUGH A RESISTANCE OF 100 OHMS.

The curve of discharge for the same condenser under the same conditions was given in Fig. 18.

CASE III. ELECTROMOTIVE FORCE A SIMPLE HARMONIC FUNCTION OF THE TIME.

Let us now suppose the impressed E. M. F. to be a simple harmonic function of the time, as in Case III., Chap.

III., in the discussion of circuits containing resistance and self-induction; that is,

$$e = f(t) = E \sin \omega t, \tag{73}$$

where E is the amplitude, or maximum value of the E. M. F., and ω the angular velocity. By differentiation,

$$\frac{de}{dt} = f'(t) = E\,\omega \cos \omega t.$$

Substituting these values in the general equations (63) and (64), we obtain

$$i = \frac{E\,\omega}{R} \epsilon^{-\frac{t}{RC}} \int \epsilon^{+\frac{t}{RC}} \cos \omega t\, dt + c_1\, \epsilon^{-\frac{t}{RC}}. \tag{74}$$

$$q = \frac{E}{R} \epsilon^{-\frac{t}{RC}} \int \epsilon^{+\frac{t}{RC}} \sin \omega t\, dt + c_2\, \epsilon^{-\frac{t}{RC}}. \tag{75}$$

These integrals may be found by the formulæ of reduction, obtained by integrating by parts, given in equations (25) and (24).

Applying these formulæ of reduction to equations (74) and (75), they become

$$i = \frac{C^2 E R \omega}{1 + C^2 R^2 \omega^2} \left\{ \omega \sin \omega t + \frac{1}{R\,C} \cos \omega t \right\} + c_1 \epsilon^{-\frac{t}{RC}}. \tag{76}$$

$$q = \frac{C^2 E R}{1 + C^2 R^2 \omega^2} \left\{ \frac{1}{R\,C} \sin \omega t - \omega \cos \omega t \right\} + c_2 \epsilon^{-\frac{t}{RC}}. \tag{77}$$

These equations (76) and (77) may be simplified by the trigonometric formula

$$A \sin \theta + B \cos \theta = \sqrt{A^2 + B^2}\, \sin \left\{ \theta + \tan^{-1} \frac{B}{A} \right\}. \tag{27}$$

By the application of this formula to (76) and (77) we have the complete solutions of the differential equations, namely,

$$(78)\quad i = \frac{E}{\sqrt{R^2 + \frac{1}{C^2\omega^2}}} \sin\left\{\omega t + \tan^{-1}\frac{1}{CR\omega}\right\} + c_1 \epsilon^{-\frac{t}{RC}},$$

and

$$q = \frac{E}{\omega\sqrt{R^2 + \frac{1}{C^2\omega^2}}} \sin\{\omega t - \tan^{-1} CR\omega\} + c_2 \epsilon^{-\frac{t}{RC}}.$$

The last equation is equivalent to

$$(79)\quad q = \frac{-E}{\omega\sqrt{R^2 + \frac{1}{C^2\omega^2}}} \cos\left\{\omega t + \tan^{-1}\frac{1}{CR\omega}\right\} + c_2 \epsilon^{-\frac{t}{RC}}.$$

These equations (78) and (79) are the complete solutions, expressed in their simplest forms. It will be noticed that the differential of (79) is (78), according to the relation $dq = i\,dt$. It was not necessary to carry both equations through together, as one may be directly derived from the other by integration or differentiation. It is thought it may add interest to the case if we have the two to compare, so that any differences that exist become more apparent.

After a very short time the last term of each of these equations, containing the arbitrary constant of integration, becomes inappreciably small and may be neglected. Then it is seen that the current and charge are both harmonic functions of the time; but the current, instead of lagging behind the impressed E. M. F., as it did in the case where there was self-induction in the circuit, advances ahead of it by an angle whose tangent is $\frac{1}{CR\omega}$. When the capacity C is infinite (and there is no condenser in the circuit, as ex-

plained on page 67) the tangent $\frac{1}{CR\omega}$ is zero, and the current is in phase with the E. M. F. When the condenser alone is in circuit, so that the resistance is negligible, $\frac{1}{CR\omega}$ becomes very large and the angle of advance is nearly 90°. $\frac{1}{C\omega}$ has been termed the reactance.

The equation of the current then becomes

$$(80) \qquad i = CE\omega \sin(\omega t + 90°),$$

and of charge

$$(81) \qquad q = -Q\cos(\omega t + 90°);$$

and the charge will always be a maximum when the current is zero and *vice versa,* as the cosine is a maximum when the sine is zero.

When the sin $\left\{\omega t + \tan^{-1}\frac{1}{CR\omega}\right\}$ becomes unity the current has its maximum value I, and

$$(82) \qquad I = \frac{E}{\sqrt{R^2 + \frac{1}{C^2\omega^2}}}.$$

The radical $\sqrt{R^2 + \frac{1}{C^2\omega^2}}$ is the apparent resistance of the circuit; and, upon comparing with equation (29), we see that it corresponds to the radical $\sqrt{R^2 + L^2\omega^2}$, which has been called the "impedance" of the circuit, in the case where there is self-induction and resistance only.

Case IV. Any Periodic Electromotive Force which is not Harmonic.

If the impressed electromotive force is any periodic function whatsoever of the time, then—as was mentioned

in the discussion of circuits containing self-induction—this E. M. F. may be expressed, according to a theorem due to Fourier, as the sum of terms of the form

$$E \sin (b\,\omega\,t + \theta).$$

Thus,

$$(83) \qquad e = \sum_{E,\, b,\, \theta.} E \sin (b\,\omega\,t + \theta)$$

may represent any electromotive force whatsoever, where E, b, and θ have n different values corresponding to n terms of the sum. As was previously shown in the case of self-induction, each term of the E. M. F. impressed gives rise to a corresponding term in the resultant current equation of the form

$$\frac{E}{\sqrt{R^2 + \dfrac{1}{C^2\, b^2\, \omega^2}}} \sin \left\{ b\,\omega\,t + \theta + \tan^{-1} \frac{1}{C\,R\,b\,\omega} \right\},$$

where E, b, and θ have values equal to their values in the corresponding term of the E. M. F. equation.

The expression for current, then, when (83) is the impressed E. M. F., is

$$(84) \quad i = \sum_{E,\, b,\, \theta.} \frac{E}{\sqrt{R^2 + \dfrac{1}{C^2\, b^2\, \omega^2}}} \sin \left\{ b\,\omega\,t + \theta + \tan^{-1} \frac{1}{C\,R\,b\,\omega} \right\} + c\,\epsilon^{-\frac{t}{RC}}.$$

This gives the general solution for the current in a simple circuit containing resistance and capacity, and any impressed E. M. F. The discussion of this general solution will be deferred until circuits containing resistance, self-induction, and capacity have been considered.

CHAPTER VI.

CIRCUITS CONTAINING RESISTANCE, SELF INDUCTION, AND CAPACITY. GENERAL SOLUTION.

CONTENTS.—Equation of energy in terms of e, i, and t; in terms of e, q, and t. Equation of E. M. F.'s in terms of e, i, and t; in terms of e, q, and t. Equations transformed for solving in terms of i and t; in terms of q and t. Complete solution for i in terms of t; complete solution for q in terms of t. Four cases will be considered: I. $e = f(t) = 0$; II. $e = f(t) = E$; III. $e = f(t) = E \sin \omega t$; IV. $e = f(t) = \Sigma E \sin (b \omega t + \theta)$.

IN the preceding chapters the formation of the differential equations for circuits containing resistance and self-induction alone, and resistance and capacity alone, has been discussed, and the solution of these differential equations obtained and discussed for these two particular cases. It is now proposed to consider a circuit containing all three, resistance, self-induction, and capacity, in series, and in the present chapter to derive from the differential equations two general solutions which express, respectively, the current flowing in the circuit and the charge of electricity in the condenser, at any moment, when the circuit is subjected to any impressed electromotive force whatsoever. The succeeding five chapters of Part I will then be devoted to a discussion of these general equations, now to be obtained, and their application to various particular cases of impressed electromotive forces.

The differential equation of energy for a circuit containing all three, resistance, self-induction, and capacity, may

be written at once, since we have already derived expressions which represent the energy used in heating the conductor [see equation (4)], in creating the magnetic field around the conductor [see equation (6)], and in charging the condenser [see equation (53)].

The equation of energy is

$$e\,i\,dt = R\,i^2\,dt + L\,i\frac{d\,i}{d\,t}\,d\,t + \frac{i\,d\,t\int i\,d\,t}{C}. \tag{85}$$

The first member of this differential equation $e\,i\,d\,t$ represents the total energy supplied to the circuit in the time $d\,t$. A part of this energy represented by $R\,i^2\,d\,t$ is used in heating the conductor. A second part $L\,i\frac{d\,i}{d\,t}\,d\,t$ is expended in creating a magnetic field in the space surrounding the conductor. A third part, represented by $\frac{i\,d\,t\int i\,d\,t}{C}$, is expended in charging the condenser. Equation (85) is the general differential equation of energy, in terms of the current which flows in the circuit, the E. M. F. which drives the current, and the time, for a circuit containing resistance, self-induction, and capacity in series.

This equation of energy may be expressed as a differential equation in terms of the quantity of electricity in the condenser, that is, the charge of the condenser, the E. M. F., and the time, by means of the relation $d\,q = i\,d\,t$, or $q = \int i\,d\,t$. On substituting in (85) $i = \frac{d\,q}{d\,t}$, we have

$$e\frac{d\,q}{d\,t}\,d\,t = R\left\{\frac{d\,q}{d\,t}\right\}^2 d\,t + L\frac{d^2\,q}{d\,t^2}\frac{d\,q}{d\,t}\,d\,t + \frac{q}{C}\frac{dq}{dt}\,d\,t. \tag{86}$$

Each term in this equation is equal to the corresponding term in equation (85), since it is obtained by direct

substitution. The first member, $e \frac{dq}{dt} dt$, is the total energy supplied to the circuit, and the three terms of the second member represent the three ways in which this energy is expended, viz., in heat, creating the field, and charging the condenser.

If equation (85) is divided through by $i\,dt$, it becomes an equation of E. M. F.'s, thus:

$$(87) \qquad e = Ri + L\frac{di}{dt} + \frac{\int i\,dt}{C}.$$

If equation (86) is divided through by $\frac{dq}{dt} dt$, it likewise becomes an equation of E. M. F.'s, thus:

$$(88) \qquad e = R\frac{dq}{dt} + L\frac{d^2 q}{dt^2} + \frac{q}{C}.$$

These are equations of E. M. F.'s: equation (87) in terms of current, E. M. F., and time; and equation (88) in terms of the charge of the condenser, E. M. F., and time. Each term in (88) is equal to the corresponding term in (87). The first member, e, is the E. M. F. impressed upon the circuit. That part of e necessary to overcome the resistance is Ri, or $R\frac{dq}{dt}$. That part of e necessary to overcome the counter E. M. F. of self-induction is $L\frac{di}{dt}$, or $L\frac{d^2 q}{dt^2}$. The third part of e, necessary to overcome the counter E. M. F. of the condenser, is $\frac{\int idt}{C}$, or $\frac{q}{C}$.

These differential equations may be written in forms more convenient for solving. Differentiating equation (87)

with regard to t, to free it from the integral sign, we obtain

$$\frac{d^2 i}{d t^2}+\frac{R}{L}\frac{d i}{d t}+\frac{i}{L C}=\frac{1}{L}\frac{d e}{d t}. \tag{89}$$

By transposition (88) becomes

$$\frac{d^2 q}{d t^2}+\frac{R}{L}\frac{d q}{d t}+\frac{q}{L C}=\frac{e}{L}. \tag{90}$$

We know that the impressed E. M. F. has one value at one particular time and is therefore a single-valued function of the time, that is, $e=f(t)$. When we introduce this relation into (89) and (90), the general solution of each of these equations may be readily obtained. The solution of equation (89) will give the value of the current at any time, and the solution of equation (90) will give the value of the charge of the condenser at any time.

If $e=f(t)$, and $\frac{d e}{d t}=f'(t)$, upon substitution in (89) and (90), we have

$$\frac{d^2 i}{d t^2}+\frac{R}{L}\frac{d i}{d t}+\frac{i}{L C}=\frac{1}{L}f'(t). \tag{91}$$

$$\frac{d^2 q}{d t^2}+\frac{R}{L}\frac{d q}{d t}+\frac{q}{L C}=\frac{1}{L}f(t). \tag{92}$$

GENERAL SOLUTION FOR CURRENT AT ANY TIME.

In solving equation (91) to obtain the value of the current at any time, it is convenient to make use of the symbolic method for linear equations. (See page 101, Johnson's Differential Equations.)

$$\text{Let } D=\frac{d[\ ]}{d t}, \qquad D^2=\frac{d^2[\ ]}{d t^2}.$$

Writing (91) in symbolic form, we have

$$\left\{ D^2 + \frac{R}{L} D + \frac{1}{L C} \right\} i = \frac{1}{L} f'(t), \quad \text{or}$$

(93)
$$i = \frac{1}{L \left\{ D^2 + \frac{R}{L} D + \frac{1}{L C} \right\}} f'(t).$$

Resolving the inverse operator, $\dfrac{1}{D^2 + \frac{R}{L} D + \frac{1}{L C}}$, into partial fractions, we have the identical equation

(94)
$$\frac{1}{D^2 + \frac{R}{L} D + \frac{1}{L C}} = \frac{L C}{\sqrt{R^2 C^2 - 4 L C}}$$

$$\left\{ \frac{1}{D + \frac{R C - \sqrt{R^2 C^2 - 4 L C}}{2 L C}} - \frac{1}{D + \frac{R C + \sqrt{R^2 C^2 - 4 L C}}{2 L C}} \right\}.$$

(95) Let $T_1 = \dfrac{2 L C}{R C - \sqrt{R^2 C^2 - 4 L C}}$,

and $T_2 = \dfrac{2 L C}{R C + \sqrt{R^2 C^2 - 4 L C}}$.

Placing these values in (94), and substituting (94) in (93), we obtain

(96)
$$i = \frac{C}{\sqrt{R^2 C^2 - 4 L C}} \left\{ \frac{1}{D + \frac{1}{T_1}} f'(t) - \frac{1}{D + \frac{1}{T_2}} f'(t) \right\}.$$

Each term of equation (96), equated separately to i, forms a linear equation of the first order. This will be evident when we consider the linear equation of the first

order between the variables x and y, viz., $\frac{dy}{dx} + ay = f(x)$. When written in the symbolic form this becomes

$$(D + a)y = f(x),$$

$$\text{(97)} \qquad \text{or} \qquad y = \frac{1}{D + a} f(x).$$

The solution of this linear equation of the first order is known to be (see Johnson's Diff. Equations, page 31)

$$\text{(98)} \qquad y = \epsilon^{-ax} \int \epsilon^{ax} f(x)\, dx + c\,\epsilon^{-ax}.$$

Here c is the arbitrary constant of integration, and none other must be added when the integration is performed.

By equating (97) and (98), we have

$$\frac{1}{D + a} f(x) = \epsilon^{-ax} \int \epsilon^{ax} f(x)\, dx + c\,\epsilon^{-ax}.$$

If we replace a, in this general formula, by the constant $\frac{1}{T_1}$, and $f(x)$ by $f'(t)$, we have

$$\frac{1}{D + \frac{1}{T_1}} f'(t) = \epsilon^{-\frac{t}{T_1}} \int \epsilon^{\frac{t}{T_1}} f'(t)\, dt + c\,\epsilon^{-\frac{t}{T_1}}.$$

But this is the value of the first term in the parenthesis of equation (96). The value of the second term in that parenthesis may be found in a similar manner, and (96) may finally be written

$$\text{(99)} \qquad i = \frac{C}{\sqrt{R^2 C^2 - 4LC}} \left\{ \epsilon^{-\frac{t}{T_1}} \int \epsilon^{\frac{t}{T_1}} f'(t)\, dt - \epsilon^{-\frac{t}{T_2}} \int \epsilon^{\frac{t}{T_2}} f'(t)\, dt \right\} + c_1 \epsilon^{-\frac{t}{T_1}} + c_2 \epsilon^{-\frac{t}{T_2}}.$$

This is the general solution of equation (91) and gives the current which flows at any time in a circuit having resistance, self-induction, and capacity.

Since the differential equation (92) for the charge becomes identical with the differential equation (91) for the current when we write $f'(t)$ instead of $f(t)$, and since f denotes any arbitrary single-valued function whatever, we may in the general solution (99) suppress the accents on the arbitrary functions and write the solution for q. Thus,

$$(100) \quad q = \frac{C}{\sqrt{R^2 C^2 - 4LC}} \left\{ \epsilon^{-\frac{t}{T_1}} \int \epsilon^{\frac{t}{T_1}} f(t)\, dt - \epsilon^{-\frac{t}{T_2}} \int \epsilon^{\frac{t}{T_2}} f(t)\, dt \right\} + c_1 \epsilon^{-\frac{t}{T_1}} + c_2 \epsilon^{-\frac{t}{T_2}}.$$

PARTICULAR ELECTROMOTIVE FORCES.

These equations, (99) and (100), express the values of the current and charge at any time, when the impressed E. M. F. is anything whatever, since f is any arbitrary single-valued function whatever.

There are four cases, covering all possible ones, which arise according to the nature of the impressed E. M. F. These are:

Case I. $e = f(t) = 0.$

Case II. $e = f(t) = E = \text{constant}.$

Case III. $e = f(t) = E \sin \omega t.$

Case IV. $e = f(t) = \sum_{E, b, \theta} E \sin (b \omega t + \theta).$

The meaning of the first assumption is that the impressed E. M. F. is to be zero at every point of time. This condition is fulfilled if we charge a condenser with some

quantity Q, and then suddenly remove the impressed E. M. F., that is, if we connect the two plates of the condenser by a conductor so as to discharge it. The impressed E. M. F. remains zero at every point of time after the removal of the source of E. M. F., and consequently satisfies the condition $e = f(t) = 0$. The solutions of the differential equations under this assumption give the current at any time flowing in the circuit, and the charge at any time remaining in the condenser, when an impressed E. M. F. is suddenly removed from the circuit. It may be any circuit whatever containing any combination of resistance, self-induction, and capacity, that is, a circuit containing R and L alone, R and C alone, or R, L, and C together. In case the circuit has R and C, or R, L, and C, the solutions will give the current i and quantity q at any time during the *discharge* of the condenser. If the circuit contains R and L alone, the solution will give the current at any time as it dies away after the removal of the E. M. F.

When we assume $e = f(t) = E =$ a constant, we mean that the E. M. F. is to be equal to E at every point of time. This condition will be fulfilled if the source of E. M. F. in any circuit is suddenly changed from one constant value to another constant value, either of which may be zero. If the circuit contains R and C, or R, L, and C, the solutions give the current flowing in the conductor and the charge of the condenser at any time after the change in the E. M. F. If the circuit contains R and L only, the solution gives the value of the current at any time as it changes to its final steady value.

The third assumption, $e = E \sin \omega t$, means that the circuit contains an impressed E. M. F. varying harmonically with the time. The solutions of the general equations for q and i show that when the impressed E. M. F. is harmonic, both the current and the charge are likewise

simple sine-functions of the time, having the same period as the E. M. F.

The fourth assumption, $e = \sum_{E, b, \theta} E \sin (b \omega t + \theta)$,—where b takes in succession any integer values,—means that the circuit contains an impressed E. M. F. which is any periodic function of the time whatsoever.

The solution and discussion of these four cases will be considered in the following chapters.

CHAPTER VII.

CIRCUITS CONTAINING RESISTANCE, SELF INDUCTION, AND CAPACITY.

CASE I. DISCHARGE.

CONTENTS :—Integral and differential equations when $e = f(t) = 0$. Sir Wm. Thomson's solution. i equation with value of T replaced. Three forms of i and q equations. To transform the i-equation to a real form when $R^2 C$ is less than $4L$. To derive the solutions from the differential equations when $R^2 C = 4L$.

Non-oscillatory Discharge.

Determination of constants. Complete solution. Value of T replaced. Current and charge curves for a particular circuit. Time of maximum current. Equation (125) applied to a circuit containing resistance and self-induction only, and to a circuit containing resistance and capacity only.

Oscillatory Discharge.

Determination of constants. Complete solution for i and q. Current and charge curves for a particular circuit.

Discharge of Condenser when $R^2 C = 4L$.

Determination of constants. Complete solutions for i and q. Figure showing method of constructing the current and charge curves. Curves for i and q in a particular circuit.

IN this chapter the case will be discussed in which the impressed electromotive force is suddenly removed from the circuit or reduced to zero; that is, $e = f(t) = 0$. When a current has been flowing in a circuit and the source of electromotive force has been suddenly removed, the current continues to flow for an appreciable time before

dying entirely away. The value of the current at any time may be ascertained by applying the general equation (99) to this particular case.

As another example, we may have a condenser or Leyden jar charged to a certain difference of potential, and the source of potential then removed. If we now connect the two plates of the condenser or coatings of the jar with a conducting wire, a current flows through the wire and the condenser is discharged. The source of potential was previously removed, and so $e = f(t) = 0$. The general equations (99) and (100) can be applied to this particular case, enabling us to ascertain the current which flows at any time in the circuit and the charge remaining in the condenser.

Since $f(t) = 0$, the first derivative is $f'(t) = 0$; and if the value $f'(t) = 0$ is substituted in the general equation (99) for current, and the value $f(t) = 0$ in equation (100) for charge, we have

$$i = c_1 \epsilon^{-\frac{t}{T_1}} + c_2 \epsilon^{-\frac{t}{T_2}}. \tag{101}$$

$$q = c_3 \epsilon^{-\frac{t}{T_1}} + c_4 \epsilon^{-\frac{t}{T_2}}. \tag{102}$$

Had the value $e = 0 = f(t)$ been substituted in the differential equation (92), and $f'(t) = 0$ in equation (91), we should have had

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0. \tag{103}$$

$$\frac{d^2 q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = 0. \tag{104}$$

It is to be noted that the form of the differential equation for i is identical with that for q. Hence their

integrals (101) and (102) have the same form, although with different arbitrary constants of integration. The solutions of the differential equations (103) and (104)—which are identical with (91) and (92) when their second members are zero—give what is called the "complementary function" (see Johnson's Differential Equations, Art. 94). The complementary function contains all the arbitrary constants of integration. The sum of the particular integral—found to satisfy equations (91) and (92) when the second member is not zero—and the complementary function gives the complete integral of the general differential equations (91) or (92).

The particular case of the *discharge* of a condenser through a circuit possessing resistance and self-induction has been fully discussed by Sir Wm. Thomson and was published as early as 1853 in the *Philosophical Magazine.* He obtained equation (102) as his result, which he showed could be expressed in two different forms, according as T_1 and T_2 are real or imaginary.

Writing equation (101) in full, by replacing the values of T_1 and T_2 given in (95), we have

$$(105) \qquad i = c_1 \epsilon^{-\frac{RC - \sqrt{R^2C^2 - 4LC}}{2LC}t} + c_2 \epsilon^{-\frac{RC + \sqrt{R^2C^2 - 4LC}}{2LC}t}.$$

If the value of $R^2 C$ is greater than $4L$, the value of i is real; but if $R^2 C$ is less than $4L$, i apparently assumes an imaginary form. It will be shown, however, that i can by a trigonometric transformation be expressed in a real form when $R^2 C$ is less than $4L$.

When $R^2 C$ is equal to $4L$ and we have the critical case, it is evident that the two terms of equation (105) may be written as one, and thus the two arbitrary constants combine into one. The complete solution, which must contain two arbitrary constants, inasmuch as it is derived from a

differential equation of the second order, cannot be readily obtained in this case from (105); but it will be directly obtained from the differential equations (103) and (104).

TO TRANSFORM EQUATION (105) TO A REAL FORM WHEN $R^2 C$ IS LESS THAN $4L$.

After factoring out the common factor $\epsilon^{-\frac{Rt}{2L}}$, we may write (105) in another form, thus:

$$(106) \quad i = \epsilon^{-\frac{Rt}{2L}} \left\{ c_1 \epsilon^{\frac{j\sqrt{4LC - R^2C^2}}{2LC}t} + c_2 \epsilon^{-\frac{j\sqrt{4LC - R^2C^2}}{2LC}t} \right\}$$

Here j is used to represent $\sqrt{-1}$. If we write

$$(107) \qquad \theta = \frac{\sqrt{4LC - R^2C^2}}{2LC} t,$$

then (106) becomes

$$(108) \qquad i = \epsilon^{-\frac{Rt}{2L}} \left\{ c_1 \epsilon^{j\theta} + c_2 \epsilon^{-j\theta} \right\}.$$

The sine and cosine may be written in exponential form* thus:

$$(109) \quad \sin\theta = \frac{\epsilon^{j\theta} - \epsilon^{-j\theta}}{2j}, \quad \text{and} \quad \cos\theta = \frac{\epsilon^{j\theta} + \epsilon^{-j\theta}}{2}.$$

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* By Maclaurin's theorem for the expansion of a function into a series, the sine and cosine may be developed into the following series:

$$(1) \qquad \sin\theta = \theta - \frac{\theta^3}{1\cdot2\cdot3} + \frac{\theta^5}{1\cdot2\cdot3\cdot4\cdot5} - \frac{\theta^7}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7} + \text{etc.}$$

$$(2) \qquad \cos\theta = 1 - \frac{\theta^2}{1\cdot2} + \frac{\theta^4}{1\cdot2\cdot3\cdot4} - \frac{\theta^6}{1\cdot2\cdot3\cdot4\cdot5\cdot6} + \text{etc.}$$

Also the development of $\epsilon^{j\theta}$ into series gives

$$(3) \quad \epsilon^{j\theta} = 1 + j\theta - \frac{\theta^2}{1\cdot2} - j\frac{\theta^3}{1\cdot2\cdot3} + \frac{\theta^4}{1\cdot2\cdot3\ 4} + j\frac{\theta^5}{1\cdot2\cdot3\cdot4\cdot5}$$
$$- \frac{\theta^6}{1\cdot2\cdot3\cdot4\cdot5\cdot6} - j\frac{\theta^7}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7} + \frac{\theta^8}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8} + \cdots$$

Therefore $\qquad \cos\theta + j\sin\theta = \epsilon^{j\theta},$

and $\qquad \cos\theta - j\sin\theta = \epsilon^{-j\theta}.$

Multiplying through by c_1 and c_2, respectively, and adding, we have

$$(110)\quad c_1\epsilon^{j\theta} + c_2\epsilon^{-j\theta} = (c_1 + c_2)\cos\theta + (c_1 - c_2)j\sin\theta.$$

If c_1 and c_2 are conjugate imaginary quantities, they may be written

$$c_1 = \frac{A + Bj}{2},$$

$$c_2 = \frac{A - Bj}{2},$$

Multiplying (1) by j and adding to (2), we find that the resulting series is identical with (3). Hence we obtain

$$(4)\qquad \cos\theta + j\sin\theta = \epsilon^{j\theta}.$$

The expansion of $\epsilon^{-j\theta}$ into series gives

$$(5)\quad \epsilon^{-j\theta} = 1 - j\theta - \frac{\theta^2}{1\cdot 2} + j\frac{\theta^3}{1\cdot 2\cdot 3} + \frac{\theta^4}{1\cdot 2\cdot 3\cdot 4} - j\frac{\theta^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \frac{\theta^6}{1.2\cdot 3\cdot 4\cdot 5\cdot 6} + \cdots$$

Multiplying (1) by j and subtracting from (2), the resulting series is identical with (5). Hence we have

$$(6)\qquad \cos\theta - j\sin\theta = \epsilon^{-j\theta}.$$

Adding equations (4) and (6) and dividing by 2, we get

$$(7)\qquad \cos\theta = \frac{\epsilon^{j\theta} + \epsilon^{-j\theta}}{2}$$

Subtracting (6) from (4) and dividing by $2j$, we have

$$(8)\qquad \sin\theta = \frac{\epsilon^{+j\theta} - \epsilon^{-j\theta}}{2j}.$$

where A and B are both real quantities. Taking the sum and difference,

$$c_1 + c_2 = A,$$

$$c_1 - c_2 = Bj;$$

and substituting these values in (110), we have

$$c_1 \epsilon^{j\theta} + c_2 \epsilon^{-j\theta} = A \cos\theta + B \sin\theta, \tag{111}$$

where c_1 and c_2 are imaginary, while A and B are real quantities. Substituting (111) in (108), we obtain

$$i = \epsilon^{-\frac{Rt}{2L}}(A \cos\theta + B \sin\theta). \tag{112}$$

By the trigonometric formula [see Chapter III, equation (27)],

$$A \cos\theta + B \sin\theta = \sqrt{A^2 + B^2} \sin\left(\theta + \tan^{-1}\frac{A}{B}\right),$$

we may finally write equation (112), after restoring the value of θ from (107), in the form

$$i = A\epsilon^{-\frac{Rt}{2L}} \sin\left\{\frac{\sqrt{4LC - R^2C^2}}{2LC}t + \Phi\right\}, \tag{113}$$

where A and Φ are the arbitrary constants of integration. Here A is not the same as in equation (112), but stands for $\sqrt{A^2 + B^2}$, and Φ stands for $\tan^{-1}\frac{A}{B}$. This equation is the equivalent of (105). It is real when (105) is imaginary and imaginary when (105) is real.

TO DERIVE THE SOLUTIONS FROM THE DIFFERENTIAL EQUATIONS WHEN $R^2C = 4L$.

If R^2C is equal to $4L$, then the differential equations (103) and (104) become

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{R^2}{4L^2} i = 0, \quad \text{and} \tag{114}$$

$$\frac{d^2 q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{R^2}{4L^2} q = 0. \tag{115}$$

Upon substituting $i = \epsilon^{mt}$, we have

$$m^2 + \frac{R}{L} m + \frac{R^2}{4L^2} = 0, \tag{116}$$

which is seen to be a perfect square as it stands, and consequently the two values of m become equal, and $m = -\frac{R}{2L}$. When there are equal roots, the solution is of the form

$$i = c_1 \epsilon^{mt} + c_2 t \epsilon^{mt}$$

(see Johnson's Diff. Equations, page 95); or, replacing m by its value, $-\frac{R}{2L}$, we have as the complete solutions

$$i = c_1 \epsilon^{-\frac{Rt}{2L}} + c_2 t \epsilon^{-\frac{Rt}{2L}}, \tag{117}$$

$$q = c' \epsilon^{-\frac{Rt}{2L}} + c'' t \epsilon^{-\frac{Rt}{2L}}. \tag{118}$$

Returning to equation (103), we may write its solution (101), the complementary function, in three different real forms, according as the value of $R^2 C$ is greater than, less than, or equal to $4L$. These forms are :

When $R^2C > 4L$,

$$(119)\quad i = c_1 \epsilon^{-\frac{RC-\sqrt{R^2C^2-4LC}}{2LC}t} + c_2 \epsilon^{-\frac{RC+\sqrt{R^2C^2-4LC}}{2LC}t}.$$

When $R^2C < 4L$,

$$(120)\quad i = A\epsilon^{-\frac{Rt}{2L}} \sin\left\{\frac{\sqrt{4LC-R^2C^2}}{2LC}t + \Phi\right\}.$$

When $R^2C = 4L$,

$$(121)\quad i = c_1 \epsilon^{-\frac{Rt}{2L}} + c_2 t \epsilon^{-\frac{Rt}{2L}}.$$

The value of the charge q given by equation (102), being of the same form as (101), may take three different forms according as R^2C is greater than, less than, or equal to $4L$; and these forms only differ from the above in the arbitrary constants, thus:

When $R^2C > 4L$,

$$(122)\quad q = c'\epsilon^{-\frac{RC-\sqrt{R^2C^2-4LC}}{2LC}t} + c''\epsilon^{-\frac{RC+\sqrt{R^2C^2-4LC}}{2LC}t}.$$

When $R^2C < 4L$,

$$(123)\quad q = A'\epsilon^{-\frac{Rt}{2L}} \sin\left\{\frac{\sqrt{4LC-R^2C^2}}{2LC}t + \Phi'\right\}.$$

When $R^2C = 4L$,

$$(124)\quad q = c'\epsilon^{-\frac{Rt}{2L}} + c'' t \epsilon^{-\frac{Rt}{2L}}.$$

The constants of integration in these equations are determined by the initial conditions imposed by the problem. For instance, if a condenser charged with a quantity Q is suddenly discharged through a circuit with resistance and self-induction, we may count the time from the moment of discharge, and thus have $q = Q$ and $i = 0$ when $t = 0$, and $q = 0$ and $i = 0$ when $t = \infty$.

NON-OSCILLATORY DISCHARGE.

Determination of Constants.—The equations (119) and (122) may be written as in (101) and (102), in terms of the time-constants T_1 and T_2 [see (95)], thus:

$$(125) \qquad i = c_1 \epsilon^{-\frac{t}{T_1}} + c_2 \epsilon^{-\frac{t}{T_2}}.$$

$$(126) \qquad q = c' \epsilon^{-\frac{t}{T_1}} + c'' \epsilon^{-\frac{t}{T_2}}.$$

The arbitrary constants c_1, c_2, c', c'' of these equations will be determined according to the conditions mentioned above, viz., when $t = 0$, $i = 0$ and $q = Q$; when $t = \infty$, $i = 0$ and $q = 0$. Substituting in (125) $i = 0$ when $t = 0$, and in (126) $q = Q$ when $t = 0$, we have

$$0 = c_1 + c_2, \quad \text{or} \quad c_1 = - c_2.$$

$$(127) \qquad Q = c' + c''.$$

Since we have the relation $dq = i\,dt$, we may differentiate (126) and write

$$i = -\frac{c'}{T_1}\epsilon^{-\frac{t}{T_1}} - \frac{c''}{T_2}\epsilon^{-\frac{t}{T_2}}.$$

Equating this and (125), we find

$$c_1 = -\frac{c'}{T_1}, \quad \text{or} \quad c' = - c_1 T_1;$$

$$c_2 = -\frac{c''}{T_2}, \quad \text{or} \quad c'' = - c_2 T_2.$$

Remembering that $c_1 = - c_2$, we may write

$$c'' = c_1 T_2.$$

Adding c' and c'',

$$c' + c'' = c_1 (T_2 - T_1) = Q. \quad \text{[See (127)]}.$$

Hence $$c_1 = \frac{Q}{T_2 - T_1};$$

$$c_2 = \frac{Q}{T_1 - T_2}$$

$$c' = \frac{Q\,T_1}{T_1 - T_2};$$

$$c'' = \frac{Q\,T_2}{T_2 - T_1}.$$

Substituting in (125) and (126) the constants c_1, c_2, c', c'', as finally determined, we have

(128) $$i = \frac{Q}{T_2 - T_1}\left\{\epsilon^{-\frac{t}{T_1}} - \epsilon^{-\frac{t}{T_2}}\right\}.$$

(129) $$q = \frac{Q}{T_1 - T_2}\left\{T_1\,\epsilon^{-\frac{t}{T_1}} - T_2\,\epsilon^{-\frac{t}{T_2}}\right\}.$$

Discussion of Non-oscillatory Discharge.—These equations give the complete solution and express the current or the charge at any time after discharge (see Fleming's "Alternate Current Transformer," Vol. I. page 376). They show that if we have the relation $R^2C > 4L$, the discharge is a gradual dying away without oscillation. Since T_1 and T_2 are each of them positive when $R^2C > 4L$ [see (95)], i or q may be represented geometrically as the difference of two decreasing logarithmic curves. To see this more clearly, the values of the time-constants T_1 and T_2 may be substituted in the coefficients of equations (128) and (129). The result is

(130) $$i = \frac{Q}{\sqrt{R^2\,C^2 - 4L\,C}}\left\{\epsilon^{-\frac{t}{T_2}} - \epsilon^{-\frac{t}{T_1}}\right\}.$$

$$(131) \quad q = \frac{Q}{2}\left\{\frac{RC}{\sqrt{R^2C^2 - 4LC}} + 1\right\}\epsilon^{-\frac{t}{T_1}} - \frac{Q}{2}\left\{\frac{RC}{\sqrt{R^2C^2 - 4LC}} - 1\right\}\epsilon^{-\frac{t}{T_2}}.$$

These equations may be more easily explained by referring to Figs. 20 and 21, which represent the plot of these equations for particular assumed values of R, L, and C. The values assumed for the constants of the circuit are

$$R = 100 \text{ ohms}, \quad L = .0016 \text{ henrys}, \quad C = 1 \text{ microfarad}.$$

By calculating the values of T_1 and T_2 [equation (95)], $T_1 = 8 \times 10^{-5}$, and $T_2 = 2 \times 10^{-5}$, the equations (130) and (131), with these particular values, become

$$(132) \qquad i = \frac{Q}{.6 \times 10^{-5}}\left\{\epsilon^{-\frac{t}{2 \times 10^{-5}}} - \epsilon^{-\frac{t}{8 \times 10^{-5}}}\right\}.$$

$$(133) \quad q = \frac{Q}{2}\left(1\tfrac{2}{3} + 1\right)\epsilon^{-\frac{t}{8 \times 10^{-5}}} - \frac{Q}{2}\left(1\tfrac{2}{3} - 1\right)\epsilon^{-\frac{t}{2 \times 10^{-5}}}.$$

If the condenser was charged to a potential of 2000 volts, the capacity being .000001 farads, the charge is .002 coulombs. Substituting this value for Q, we have

$$i = 33.33\left(\epsilon^{-\frac{t}{2 \times 10^{-5}}} - \epsilon^{-\frac{t}{8 \times 10^{-5}}}\right),$$

$$q = \frac{1}{1000}\left(1\tfrac{2}{3} + 1\right)\epsilon^{-\frac{t}{8 \times 10^{-5}}} - \frac{1}{1000}\left(1\tfrac{2}{3} - 1\right)\epsilon^{-\frac{t}{2 \times 10^{-5}}},$$

where i is in amperes and q in coulombs.

In Fig. 20, curves I. and II. represent the two component logarithmic curves, corresponding to the first and

second terms, respectively, of equation (130), whose difference gives the resultant current curve III. Curve II., corresponding to the second term, has the larger time-constant, and is therefore the more important curve. The area in-

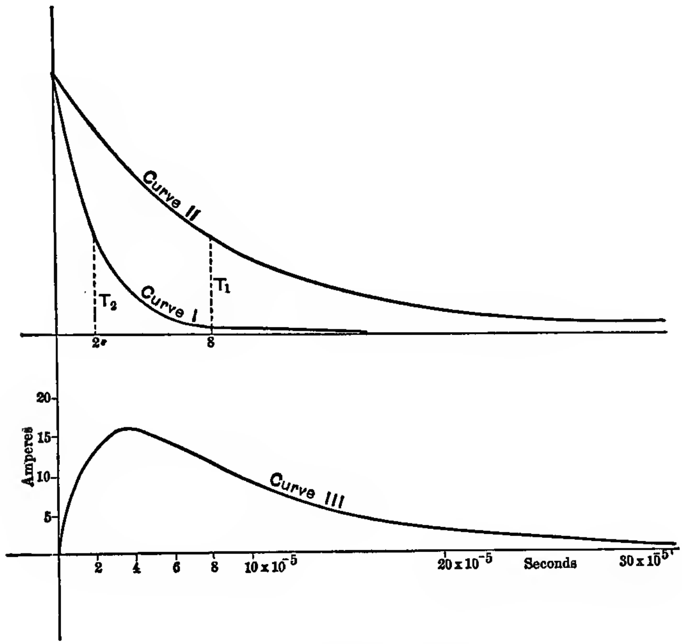

FIG. 20.—CURVE SHOWING CURRENT DURING NON-OSCILLATORY DISCHARGE OF CONDENSER WITH CAPACITY $C = 1$ MICROFARAD, THROUGH A CIRCUIT WITH RESISTANCE $R = 100$ OHMS, AND SELF-INDUCTION $L = .0016$ HENRYS, WHEN ORIGINALLY CHARGED TO A POTENTIAL OF 2000 VOLTS.

cluded between curve III. and the axis of abscissæ is equal to $\int i\,dt = Q$, and is therefore independent of the constants of the circuit through which the condenser is discharged.

The current is a maximum at a point which may be determined by differentiating equation (130) and equating the first derivative to zero in the usual manner for a maximum.

The time t_m at which the current is a maximum is thus found to be

$$t_m = \frac{LC}{\sqrt{R^2C^2 - 4LC}} \log \frac{T_1}{T_2}. \tag{134}$$

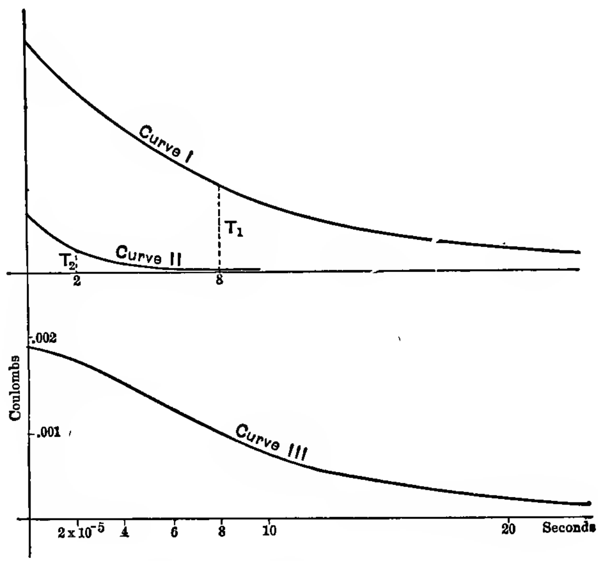

FIG. 21.—CURVE SHOWING NON-OSCILLATORY DISCHARGE OF A CONDENSER, WITH CAPACITY $C = 1$ MICROFARAD, THROUGH A CIRCUIT WITH RESISTANCE $R = 100$ OHMS, AND SELF-INDUCTION $L = .0016$ HENRYS, WHEN ORIGINALLY CHARGED TO A POTENTIAL OF 2000 VOLTS.

Substituting in (134) the particular values used in plotting Fig. 20, we find the time when the current is a maximum to be

$$t_m = 3.78 \times 10^{-5}.$$

In Fig. 21 curves I. and II. are the two component logarithmic curves, corresponding to the first and second terms, respectively, of equation (131) for charge. Curve III. is plotted by subtracting II. from I., and represents the

charge of the condenser at any time. It is noticeable that the upper curve, I., has the larger initial value, and as T_1 is larger than T_2, decreases the slower. It is therefore this curve which is the more important in determining the discharge of the condenser.

EQUATION (125) APPLIES TO A CIRCUIT CONTAINING RESISTANCE AND SELF-INDUCTION ONLY.

If there is no condenser in the circuit, as explained in Chapter IV., it is equivalent to saying that there is a condenser of infinite capacity in the circuit. Substituting $C = \infty$ in the equation (95) for the time-constants, we have

$$T_1 = \frac{2\,L\,C}{R\,C - \sqrt{R^2 C^2 - 4\,L\,C}} = \infty.$$

$$T_2 = \frac{2\,L\,C}{R\,C + \sqrt{R^2 C^2 - 4\,L\,C}} = \frac{L}{R}.$$

According to equation (101), we have the value of the current at any time

$$i = c_1\,\epsilon^{-\frac{t}{T_1}} + c_2\,\epsilon^{-\frac{t}{T_2}}.$$

Substituting in this equation the values of T_1 and T_2 above, we have

$$i = c_1 + c_2\,\epsilon^{-\frac{R\,t}{L}}.$$

When $t = 0$, $i = I$, that is, the current flowing previous to the removal of the E. M. F. This gives

$$i = I = c_1 + c_2.$$

But when $t = \infty$, $i = c_1 = 0$. Substituting these values for the constants, we have

$$i = I\,\epsilon^{-\frac{R\,t}{L}},$$

a result which is well known [see equation (18)].

EQUATION (125) APPLIES TO A CIRCUIT CONTAINING RESISTANCE AND CAPACITY ONLY.

Upon substituting $L = 0$ in the values of the time-constants T_1 and T_2 (95), the expressions become indeterminate, but can readily be evaluated by differentiating numerator and denominator, and then substituting $L = 0$ as in ordinary vanishing fractions.

$$T_1 = \frac{2LC}{RC - \sqrt{R^2C^2 - 4LC}}.$$

Differentiating numerator and denominator with respect to L, we have

$$\frac{\frac{d}{dL}(2LC)}{\frac{d}{dL}(RC - \sqrt{R^2C^2 - 4LC})} = \frac{2C}{\frac{4C}{2\sqrt{R^2C^2 - 4LC}}}$$

$$= \sqrt{R^2C^2 - 4LC}.$$

Now letting $L = 0$, we have

$$T_1 = RC. \qquad \text{Similarly,}\ T_2 = -RC.$$

Substituting these values in equations (101) and (102), we have

(135) $$i = c_1 \epsilon^{-\frac{t}{RC}} + c_2 \epsilon^{+\frac{t}{RC}}.$$

(136) $$q = c_3 \epsilon^{-\frac{t}{RC}} + c_4 \epsilon^{+\frac{t}{RC}}.$$

c_2 and c_4 must each be zero, or else when $t = \infty$ we would have $i = \infty$ and $q = \infty$. When $t = 0$, $q = Q = c_3$. By differentiating (136) and equating to (135), we have

$$i = \frac{dq}{dt} = -\frac{c_3}{RC}\epsilon^{-\frac{t}{RC}} = c_1 \epsilon^{-\frac{t}{RC}},$$

and, therefore, $c_1 = -\frac{c_3}{RC} = -\frac{Q}{RC}.$

Substituting in (135) and (136) the values found for the constants c_1, c_2, c_3, c_4, we have

$$(137) \qquad i = -\frac{Q}{RC}\epsilon^{-\frac{t}{RC}} = I\epsilon^{-\frac{t}{RC}}.$$

$$(138) \qquad q = Q\epsilon^{-\frac{t}{RC}}.$$

These are the well-known results for the case of discharge through a circuit with no self-induction [see equations (67) and (68)].

OSCILLATORY DISCHARGE.

Determination of Constants.—In the case of oscillatory discharge, the equations for current and charge at any time are

$$(120) \quad i = A\epsilon^{-\frac{Rt}{2L}}\sin\left\{\frac{\sqrt{4LC - R^2C^2}}{2LC}t + \Phi\right\}.$$

$$(123) \quad q = A'\epsilon^{-\frac{Rt}{2L}}\sin\left\{\frac{\sqrt{4LC - R^2C^2}}{2LC}t + \Phi'\right\}.$$

The arbitrary constants A, A', Φ, and Φ' will be determined according to the same conditions as those mentioned above, viz., when $t = 0$, $i = 0$ and $q = Q$; also when $t = \infty$, $i = 0$ and $q = 0$. Substituting in (120) $i = 0$ when $t = 0$, and in (123) $q = Q$ when $t = 0$, we have

$$0 = A\sin\Phi;$$

$$(139) \qquad \text{and} \qquad Q = A'\sin\Phi'.$$

Since A and $\sin\Phi$ are constants, and their product is zero, one of them must be zero. But if A is zero, i is zero for every point of time, which is impossible. Therefore

$$(140) \qquad \Phi = 0.$$

Differentiating (123) and remembering that $i = \frac{dq}{dt}$; we have

$$(141)\quad i = \frac{dq}{dt} = -\frac{A' R \epsilon^{-\frac{Rt}{2L}}}{2L} \sin\left\{\frac{\sqrt{4LC - R^2C^2}}{2LC} t + \Phi'\right\}$$

$$+ \frac{A' \epsilon^{-\frac{Rt}{2L}} \sqrt{4LC - R^2C^2}}{2LC} \cos\left\{\frac{\sqrt{4LC - R^2C^2}}{2LC} t + \Phi'\right\}.$$

Substituting $i = 0$ when $t = 0$,

$$0 = -R \sin \Phi' + \frac{\sqrt{4LC - R^2C^2}}{C} \cos \Phi'.$$

(142) Hence $\Phi' = \tan^{-1} \frac{\sqrt{4LC - R^2C^2}}{RC}$.

By (139), $A' = \frac{Q}{\sin \Phi'}$. And, by (142),

$$(143)\quad A' = \frac{Q}{\sin \tan^{-1} \frac{\sqrt{4LC - R^2C^2}}{RC}} = \frac{Q}{\sqrt{1 - \frac{R^2C^2}{4LC}}}.$$

To determine the constant A, transform (141) by the formula (27) so as to write it in terms of a sine only. The coefficient of the sine in the equation as transformed will be

$$A' \sqrt{\frac{1}{LC}}.$$

Since equations (141) and (120) are each equations for i, we may equate the coefficients of the sine, and have

$$A = \frac{A'}{\sqrt{LC}}.$$

And, by (143),

$$(144)\quad A = \frac{2Q}{\sqrt{4LC - R^2C^2}}.$$

Substituting A and Φ, as determined in (144) and (140), in equation (120), and substituting A' and Φ', as determined in (143) and (142), in equation (123), we have

$$(145)\quad i = \frac{2\,Q}{\sqrt{4\,L\,C - R^2C^2}}\,\epsilon^{-\frac{Rt}{2L}}\sin\left\{\frac{\sqrt{4\,L\,C - R^2C^2}}{2\,L\,C}\,t\right\}.$$

$$(146)\quad q = \frac{2\,Q\sqrt{L\,C}}{\sqrt{4\,L\,C - R^2C^2}}\,\epsilon^{-\frac{Rt}{2L}}$$

$$\sin\left\{\frac{\sqrt{4\,L\,C - R^2C^2}}{2\,L\,C}\,t + \tan^{-1}\frac{\sqrt{4\,L\,C - R^2C^2}}{R\,C}\right\}.$$

Discussion of Oscillatory Discharge.—These equations may be more readily understood by referring to Fig. 22, in which curves showing the current and charge, according to these equations, are drawn for the discharge of a condenser for particular values of R, L, and C, assumed. The particular constants assumed are

$R = 100$ ohms, $L = .0125$ henrys, $C = 1$ microfarad.

If the condenser be originally charged to 2000 volts, $Q = .002$ coulombs. On substituting these values in (145) and (146), the equations for current and charge become

$$i = 20\,\epsilon^{-4000\,t}\sin 8000\,t,\ \text{ and}$$

$$q = .00224\,\epsilon^{-4000\,t}\sin\ (8000\,t + \tan^{-1} 2),$$

where i is in amperes and q in coulombs. Curve I., representing the current, is a sine-curve with an amplitude decreasing according to the logarithmic curve $20\,\epsilon^{-4000\,t}$. The period is $\frac{2\,\pi}{8000} = .000785$ seconds; that is, there are 1275 complete oscillations per second. A very few oscillations are sufficient for a complete discharge.

The charge at any time is shown in curve II., which is likewise a sine-curve with an amplitude decreasing accord-

ing to the logarithmic curve, in this case $.00224\,\epsilon^{-4000\,t}$. The scale in Fig. 22 is such that the same logarithmic curve is an envelope for the current and the charge curve. The periods of the two are the same, but the curves differ in

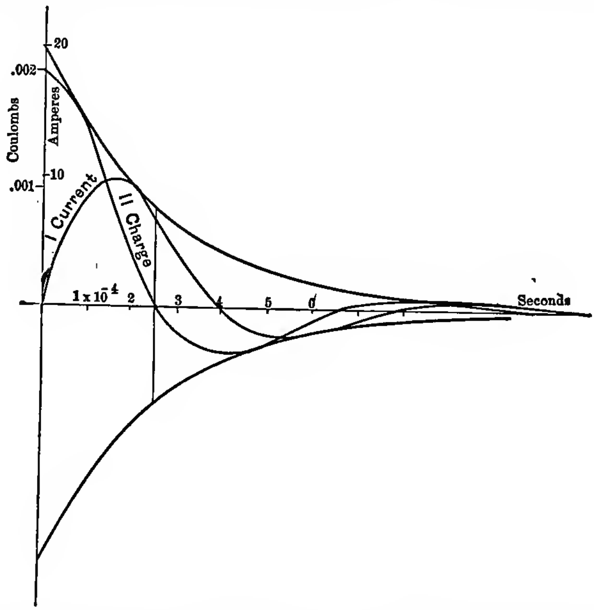

FIG. 22.—OSCILLATORY DISCHARGE OF A CONDENSER WITH CAPACITY $C = 1$ MICROFARAD, THROUGH A CIRCUIT WITH RESISTANCE $R =$ 100 OHMS, AND SELF-INDUCTION $= L$ .0125 HENRYS, WHEN ORIGINALLY CHARGED TO A POTENTIAL OF 2000 VOLTS.

phase by an angle $= \tan^{-1} 2$, that is, the charge is ahead of the current by an angle of advance of $63° \, 27'$.

DISCHARGE OF THE CONDENSER WHEN $R^2 C = 4\,L$.

Determination of Constants.—This is the critical case when the discharge is just non-oscillatory. The equations for the current and charge, as previously determined, are

$$(121) \qquad i = c_1\,\epsilon^{-\frac{Rt}{2L}} + c_2\,t\,\epsilon^{-\frac{Rt}{2L}}.$$

$$(124) \qquad q = c'\,\epsilon^{-\frac{Rt}{2L}} + c''\,t\,\epsilon^{-\frac{Rt}{2L}}.$$

The arbitrary constants of integration, c_1, c_2, c', c'', of these equations will be determined by the same conditions as in the previous cases, namely, when $t=0$, $i=0$ and $q=Q$. Equations (121) and (124) then become

$$0 = c_1, \tag{147}$$
$$Q = c'.$$

Differentiating equation (124) and substituting Q for c', we obtain

$$i = \frac{dq}{dt} = \left(-\frac{QR}{2L} + c'' - \frac{c''Rt}{2L}\right)\epsilon^{-\frac{Rt}{2L}}. \tag{148}$$

But when $t=0$, $i=0$; therefore

$$c'' = \frac{QR}{2L}. \tag{149}$$

Equating equations (121) and (148) and replacing the values for the constants given in (147) and (149), we have

$$c_2 = -\frac{QR^2}{4L^2}. \tag{150}$$

Substituting for Q its value EC, and for R^2 its equivalent, in this particular case $\frac{4L}{C}$, (150) becomes

$$c_2 = -\frac{E}{L}.$$

Having thus determined the values of the arbitrary constants, the equations for current and charge may be written

$$i = -\frac{E}{L}t\epsilon^{-\frac{Rt}{2L}}. \tag{151}$$

$$q = \left(1 + \frac{Rt}{2L}\right)Q\epsilon^{-\frac{Rt}{2L}}. \tag{152}$$

Discussion of Discharge when $R^2C = 4L$.—These equations give the value of the current and charge at any time during the discharge of the condenser in the case

where the discharge is just non-oscillatory. This case is often called the case of quickest discharge, as was pointed out by Dr. W. E. Sumpner in the *Philosophical Magazine*, and afterwards discussed by Dr. Oliver Lodge in the *Electrician* for May 18, 1888.

The curve representing the plot of the equation (151) for the current may be drawn as indicated in Fig. 23. A

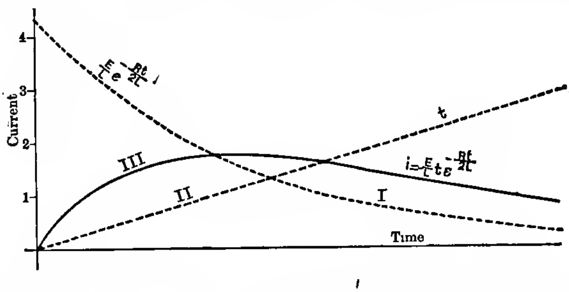

FIG. 23.—SHOWING METHOD OF CONSTRUCTING THE CURRENT CURVE IN THE CASE WHERE $R^2 C = 4 L$.

logarithmic curve I., having its initial value $\frac{E}{L}$ and time-constant $\frac{2L}{R}$, is drawn to represent the equation as it would be with t omitted from the coefficient; and each ordinate is then multiplied by the ordinate of a straight line II., which passes through the origin and represents the uniform increase of the time t. The product of the ordinates of curves I. and II. at each point gives the ordinate of the current curve III. at that point. When actual values of R, L, and C are assumed, it is found to be difficult to represent these curves to scale, so that Fig. 23 is shown simply as an illustration of the method of constructing the current curve. Curve I., Fig. 25, represents the current in an actual case where $R = 100$ ohms, $L = 2.5$ henrys, and $C = 1000$ microfarads, the condenser being originally charged to a potential of 2000 volts.

The method of constructing the curve, showing the

charge left in the condenser at any time, is given in Fig. 24, and is similar to the method just shown for constructing

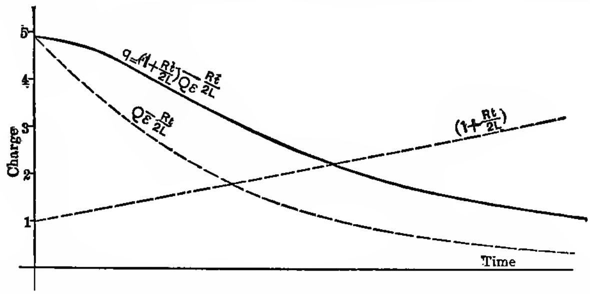

FIG. 24.—SHOWING THE METHOD OF CONSTRUCTING THE CURVE REPRESENTING THE CHARGE LEFT IN THE CONDENSER AT ANY TIME AFTER DISCHARGE.

the current curve. The difference is that the straight line passes through a point one unit above the origin, on the vertical axis, instead of through the origin as before.

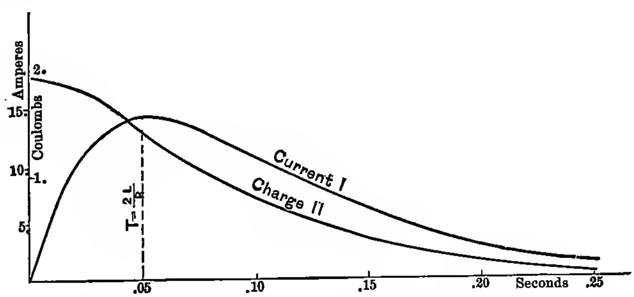

FIG. 25.—JUST NON-OSCILLATORY DISCHARGE OF A CONDENSER WITH CAPACITY $C = 1000$ MICROFARADS, THROUGH A CIRCUIT WITH RESISTANCE $R = 100$ OHMS, AND SELF-INDUCTION $L = 2.5$ HENRYS.

The logarithmic curve has the initial value Q and a time-constant $\frac{2L}{R}$. The curve showing the charge for the actual case where $R = 100$ ohms, $L = 2.5$ henrys, and $C = 1000$ microfarads, the condenser being originally charged to a potential of 2000 volts, is represented by curve II., Fig. 25.

CHAPTER VIII.

CIRCUITS CONTAINING RESISTANCE, SELF INDUCTION, AND CAPACITY.

Case II. Charge.

Contents:—Differential equations with $e = f(t) = E$. Solution of these equations. Solution from the general integral equation.. Three forms of i and q equations.

Non-oscillatory Charging.

Determination of constants. Complete solutions for i and q with constants determined. Curves for i and q in a particular circuit. Equation (101) applied to a circuit containing resistance and self-induction only; also to a circuit containing resistance and capacity only.

Oscillatory Charging.

Determination of constants. Complete solutions for i and q with constants determined. Curves for i and q in a particular circuit.

Charge of the Condenser when $R^2C = 4L$.

Determination of constants. Complete solutions for i and q with constants determined. Curves for i and q in a particular circuit.

The E. M. F., instead of being zero, as in Case I., is assumed to be a constant E, and $e = f(t) = E$. This is the case when an E. M. F. is suddenly changed from one constant value to another constant value in a circuit, and it includes Case I. as a particular case, since E may be zero. Since e is a constant, the first derivative of e is zero, and,

therefore, $f'(t) = 0$. Substituting these values of $f(t)$ and $f'(t)$ in the differential equations (89) and (90), they become

$$\frac{d^2 i}{d t^2} + \frac{R}{L}\frac{d i}{d t} + \frac{i}{L C} = 0, \tag{153}$$

(154) and $$\frac{d^2 q}{d t^2} + \frac{R}{L}\frac{d q}{d t} + \frac{q}{L C} = \frac{E}{L}.$$

It is seen that the equation for current (153) is identical with that of the previous case (103), while the equation for charge (154) has its second member equal to $\frac{E}{L}$, a constant, instead of being zero as in equation (104). By substituting a new variable, $q' = q - E\,C$, this equation may be transformed into one having its second member zero, thus:

$$\frac{d^2 q'}{d t^2} + \frac{R}{L}\frac{d q'}{d t} + \frac{q'}{L C} = 0. \tag{155}$$

The solutions of (153) and (155) are, as in the previous case,

$$i = c_1 \epsilon^{-\frac{t}{T_1}} + c_2 \epsilon^{-\frac{t}{T_2}}. \tag{101}$$

$$q' = c_3 \epsilon^{-\frac{t}{T_1}} + c_4 \epsilon^{-\frac{t}{T_2}}.$$

Replacing the value of q', and remembering $Q = E\,C =$ the final charge, we may write

$$q = Q + c_3 \epsilon^{-\frac{t}{T_1}} + c_4 \epsilon^{-\frac{t}{T_2}}. \tag{156}$$

These equations, (101) and (156), for current and charge might have been obtained directly from the general solutions (99) and (100) by substituting $f(t) = E$, and $f'(t) = 0$. Upon substituting $f'(t) = 0$ in (99), we obtain (101) directly, and upon substituting $f(t) = E$ in (100), we have

$$q = \frac{E\,C}{\sqrt{R^2 C^2 - 4\,L\,C}}(T_1 - T_2) + c_3 \epsilon^{-\frac{t}{T_1}} + c_4 \epsilon^{-\frac{t}{T_2}}.$$

But, by the values of T_1 and T_2 in (95), we find that $T_1 - T_2 = \sqrt{R^2C^2 - 4LC}$; and hence this equation is identical with (156), as $Q = EC$.

As in Case I, where $f(t) = 0$, the equations just obtained for current (101) and charge (156), when $f(t) = E$, assume three forms.

When $R^2C > 4L$,

(157) $$i = c_1 \epsilon^{-\frac{t}{T_1}} + c_2 \epsilon^{-\frac{t}{T_2}}.$$

(158) $$q = Q + c' \epsilon^{-\frac{t}{T_1}} + c'' \epsilon^{-\frac{t}{T_2}}.$$

When $R^2 C < 4L$,

(159) $$i = A\epsilon^{-\frac{Rt}{2L}} \sin\left\{\frac{\sqrt{4LC - R^2C^2}}{2LC}t + \Phi\right\}.$$

(160) $$q = Q + A'\epsilon^{-\frac{Rt}{2L}} \sin\left\{\frac{\sqrt{4LC - R^2C^2}}{2LC}t + \Phi'\right\}.$$

When $R^2C = 4L$,

(161) $$i = c_1 \epsilon^{-\frac{Rt}{2L}} + c_2 t \epsilon^{-\frac{Rt}{2L}}.$$

(162) $$q = Q + c'\epsilon^{-\frac{Rt}{2L}} + c'' t \epsilon^{-\frac{Rt}{2L}}.$$

The constants of integration must be determined by the conditions of the problem as to the previous state of the circuit, the changes made, and the final state.

NON-OSCILLATORY CHARGING.

Determination of Constants.—The constants c_1, c_2, c', and c'' of equations (157) and (158) will be determined by the following conditions:

When $t = 0$, $i = 0$ and $q = Q_0$.

When $t = \infty$, $i = 0$ and $q = Q$.

This means that the condenser is suddenly charged or discharged from the initial charge Q_0 to the final charge Q. Determining the constants by the same method as in Case I., we find that

$$c_1 = \frac{Q_0 - Q}{T_2 - T_1},$$

$$c_2 = \frac{Q_0 - Q}{T_1 - T_2},$$

$$c' = \frac{(Q_0 - Q)\,T_1}{T_1 - T_2},$$

$$c'' = \frac{(Q_0 - Q)\,T_2}{T_2 - T_1}.$$

Substituting in (157) and (158) the values of the constants just determined, we have

$$(163) \qquad i = \frac{Q_0 - Q}{T_2 - T_1}\left\{ \epsilon^{-\frac{t}{T_1}} - \epsilon^{-\frac{t}{T_2}} \right\}.$$

$$(164) \qquad q = Q + \frac{Q_0 - Q}{T_1 - T_2}\left\{ T_1\,\epsilon^{-\frac{t}{T_1}} - T_2\,\epsilon^{-\frac{t}{T_2}} \right\}.$$

For Q_0, the original charge, we may write CE_0, and for Q the final charge, we may write CE.

These equations give the value of the current and charge at any time after the change of E. M. F. from E_0 to E in a circuit with $R^2C > 4L$. As the equations now stand in their general form, they hold true for either total or partial charge or discharge according to the values of E_0 and E, and consequently Q_0 and Q, assumed. If the final charge is $Q = 0$, we have the case of complete discharge and the equations take the form of (128) and (129). If the original charge $Q_0 = 0$, we have the case of charge from zero to Q.

Discussion of Non-oscillatory Charge.—These equations will perhaps be better understood by referring to Fig. 26, which represents the equations with particular values

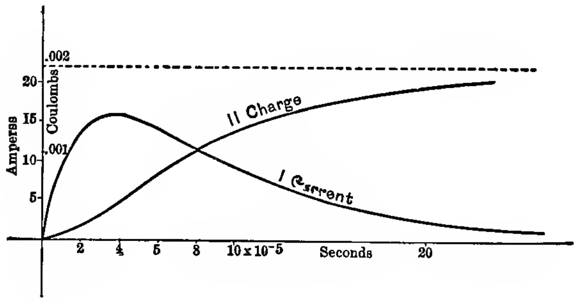

FIG. 26.—NON-OSCILLATORY DISCHARGE OF A CONDENSER WITH CAPACITY $C = 1$ MICROFARAD, THROUGH A CIRCUIT WITH RESISTANCE $R = 100$ OHMS AND SELF-INDUCTION $L = .0016$ HENRYS WHEN SUBJECTED TO A POTENTIAL OF 2000 VOLTS.

assumed. These values are the same as in the preceding case, namely, $R = 100$ ohms, $C = 1$ microfarad, $L = .0016$ henrys. The condenser originally had no charge, and when charged to a potential of 2000 volts, has a charge of .002 coulombs. The current curve I., Fig. 26, is identical with curve III., Fig. 20, which represents the current during discharge. Curve II. representing the charge is the same as curve III., Fig. 21, inverted and plotted downwards from the horizontal line $Q = .002$. It is noticeable that the ordinates of curve I., expressing the current, are proportional to the tangents of the angle of inclination of curve II. at every point, since the current $i = \frac{dq}{dt}$, and $\frac{dq}{dt}$ is the tangent of the angle of inclination of the curve of charge II. It is seen that the point of inflection on curve II. comes at the maximum value of the current curve I., as the tangent is a maximum at this point. Indeed, curve I. might be coustructed geometrically simply from the foregoing consideration.

EQUATION (101) APPLIES TO A CIRCUIT CONTAINING RESISTANCE AND SELF INDUCTION ONLY, IN THE CASE OF THE ESTABLISHMENT OF A CURRENT UPON INSERTING AN E. M. F.

In this case there is no condenser in the circuit, that is, the capacity is infinite. Substituting $C = \infty$ in the values of the time-constants (95), we have $T_1 = \infty$, $T_2 = \frac{L}{R}$, as in Case I., where the current dies away after the removal of the E. M. F. Substituting these values in (101), we have

$$i = c_1 + c_2 \epsilon^{-\frac{Rt}{L}}.$$

$$\text{When } t = 0, \quad i = 0 = c_1 + c_2.$$

$$\text{When } t = \infty, \quad i = I = c_1. \quad \therefore c_2 = -I.$$

Substituting these values for the constants c_1 and c_2, we have

$$i = I\left(1 - \epsilon^{-\frac{Rt}{L}}\right).$$

I is the final steady value of the current, and is equal to $\frac{E}{R}$; hence

$$i = \frac{E}{R}\left(1 - \epsilon^{-\frac{Rt}{L}}\right), \tag{21}$$

which is the well-known expression for the establishment of a current in a circuit with self-induction [see equation (21), Chap. III.].

EQUATION (156) APPLIES TO A CIRCUIT CONTAINING RESISTANCE AND CAPACITY ONLY IN THE CASE OF CHARGING A CONDENSER.

Upon substituting $L = 0$ in the values of the time-constants T_1 and T_2, the expressions become indeterminate, but

can be evaluated as before by differentiation of numerator and denominator before substituting $L = 0$. We thus find the values, when $L = 0$,

$$T_1 = R\,C. \qquad T_2 = -\,R\,C.$$

Substituting these values in the equations of current (101) and charge (156), we have

(165) $$i = c_1\,\epsilon^{-\frac{t}{RC}} + c_2\,\epsilon^{+\frac{t}{RC}}.$$

(166) $$q = Q + c_3\,\epsilon^{-\frac{t}{RC}} + c_4\,\epsilon^{+\frac{t}{RC}}.$$

Q_0 is the previous charge of the condenser, and Q the final charge. The constants c_2, c_4 must be zero, or else when $t = \infty$ we would have $i = \infty$, $q = \infty$. When $t = 0$, equation (166) becomes

$$Q_0 = Q + c_3. \qquad \therefore c_3 = Q_0 - Q.$$

By differentiating (166) and equating to (165), we have

$$i = \frac{d\,q}{d\,t} = -\,\frac{c_3}{R\,C}\,\epsilon^{-\frac{t}{RC}} = c_1\,\epsilon^{-\frac{t}{RC}}.$$

Therefore $$c_1 = -\,\frac{c_3}{R\,C} = -\,\frac{Q_0 - Q}{R\,C}.$$

Substituting in (165) and (166) the values for the constants c_1, c_2, c_3, c_4, as determined,

(167) $$i = -\,\frac{Q_0 - Q}{R\,C}\,\epsilon^{-\frac{t}{RC}}.$$

(168) $$q = Q + (Q_0 - Q)\epsilon^{-\frac{t}{RC}}.$$

These equations are true for the charge or discharge from Q_0 to Q, through a resistance with no self-induction. When the final charge Q is zero, we have the case of complete discharge, and the equations become the same as (137)

and (138). When the original charge Q_0 is zero, we have the case of charging from zero to Q, and equations (167) and (168) become

$$i = \frac{Q}{RC} \epsilon^{-\frac{t}{RC}}.$$

$$q = Q\left(1 - \epsilon^{-\frac{t}{RC}}\right).$$

These equations are identical with (72) and (71), already obtained in Chap. V. It is noticeable that the current equation is the same as that for discharge equation (137), and that the charge equation is analogous to that in the case of the establishment of the current in a circuit with resistance and self-induction, equation (21).

OSCILLATORY CHARGING.

Determination of Constants.—The constants A, A', Φ, and Φ' in equations (159) and (160) will be determined by the same conditions as before, namely,

$$\text{When } t = 0, \quad i = 0 \quad \text{and} \quad q = Q_0.$$

$$\text{When } t = \infty, \quad i = 0 \quad \text{and} \quad q = Q.$$

The meaning of this supposition is the same as in the preceding case, namely, that the condenser is suddenly charged or discharged from the initial charge Q_0 to the final charge Q. The constants, determined by the same method as in Case I., are

$$A = \frac{2(Q_0 - Q)}{\sqrt{4LC - R^2C^2}}.$$

$$A' = \frac{2(Q_0 - Q)\sqrt{LC}}{\sqrt{4LC - R^2C^2}}.$$

$$\Phi = 0.$$

$$\Phi' = \tan^{-1}\frac{\sqrt{4LC - R^2C^2}}{RC}.$$

With the constants thus determined, equations (159) and (160) become

$$(169) \quad i = \frac{2(Q_0 - Q)}{\sqrt{4LC - R^2C^2}} \epsilon^{-\frac{Rt}{2L}} \sin \frac{\sqrt{4LC - R^2C^2}}{2LC} t.$$

$$(170) \quad q = Q + \frac{2(Q_0 - Q)\sqrt{LC}}{\sqrt{4LC - R^2C^2}} \epsilon^{-\frac{Rt}{2L}}$$

$$\sin \left\{ \frac{\sqrt{4LC - R^2C^2}}{2LC} t + \tan^{-1} \frac{\sqrt{4LC - R^2C^2}}{RC} \right\}.$$

We may write CE_0 for the original charge Q_0, and CE for the final charge Q.

Discussion of Oscillatory Charge.—These equations give the value of the current and charge at any time after the change of the electromotive force from E_0 to E in a circuit with $R^2 C < 4L$. As the equations now stand in their general form, they are true for either total or partial charge or discharge, according to the values assigned to Q_0 and Q. If the final charge Q is zero, we have the case of complete discharge, and the equations take the form of (145) and (146). When Q is less than Q_0, we have partial discharge; if Q is greater than Q_0, we have partial charging. If the original charge $Q_0 = 0$, we have the case of charge from zero to Q.

Fig. 27 illustrates the case of oscillatory charge through a circuit having the same constants as those of Fig. 22. The current curve I. is the same as that in Fig. 22, and the charge, represented by curve II., is the same as in that figure, but inverted and plotted from the horizontal line $Q = .002$. It is seen that in charging the condenser, the charge rises at first higher than its final value. and then oscillates about that final value until it has become steady.

CHARGE OF THE CONDENSER WHEN $R^2 C = 4 L$.

Determination of Constants.—This is the critical case, where the charging is just non-oscillatory. The equations for current and charge are

$$i = c_1 \epsilon^{-\frac{Rt}{2L}} + c_2 t \epsilon^{-\frac{Rt}{2L}}. \tag{161}$$

$$q = Q + c' \epsilon^{-\frac{Rt}{2L}} + c'' t \epsilon^{-\frac{Rt}{2L}}. \tag{162}$$

The initial charge is Q_0, and the final charge Q. To determine the arbitrary constants of integration, let $t = 0$.

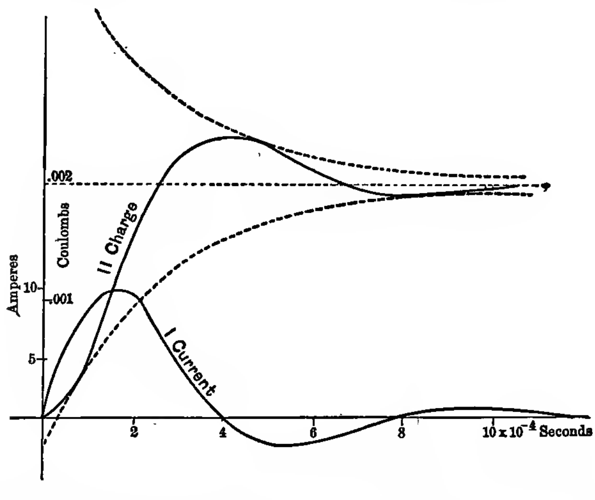

FIG. 27.—OSCILLATORY CHARGE OF A CONDENSER WITH CAPACITY $C =$ 1 MICROFARAD, THROUGH A CIRCUIT WITH RESISTANCE $R = 100$ OHMS AND SELF-INDUCTION $L = .0125$ HENRYS WHEN SUBJECTED TO A POTENTIAL OF 2000 VOLTS.

Then $i = 0$, and $q = Q_0$. Equations (161) and (162) then become

$$c_1 = 0.$$

$$c' = Q_0 - Q.$$

Differentiating equation (162) and substituting the value of c', we have

$$(171)\quad i=\frac{dq}{dt}=\left(-\frac{(Q_0-Q)R}{2L}+c''-\frac{c''Rt}{2L}\right)\epsilon^{-\frac{Rt}{2L}}.$$

When $t=0$, $i=0$; therefore

$$c''=\frac{(Q_0-Q)R}{2L}.$$

Equating equations (161) and (171), and replacing the values for c_1, c', and c'', we have

$$c_2=-(Q_0-Q)\frac{R^2}{4L^2}.$$

If E_0 and E are the initial and final potentials, respectively, we may write E_0C for Q_0, and EC for Q. Making this substitution and remembering that in this particular case $R^2=\frac{4L}{C}$, we have

$$c_2=\frac{E-E_0}{L}.$$

Replacing the values of the arbitrary constants, the equations (161) and (162) for current and charge may be written

$$(172)\qquad i=\frac{E-E_0}{L}\,t\,\epsilon^{-\frac{Rt}{2L}}.$$

$$(173)\qquad q=Q+(Q_0-Q)\left(1+\frac{Rt}{2L}\right)\epsilon^{-\frac{Rt}{2L}}.$$

Discussion of Charge when $R^2C=4L$.—The current curve in the case of charging a condenser, represented by equation (172), is the same as in the case of discharge, equation (151). It is represented in Fig. 28, curve I.,

and may be constructed by the method shown in Fig. 23. Curve II., Fig. 28, showing the charge is constructed in

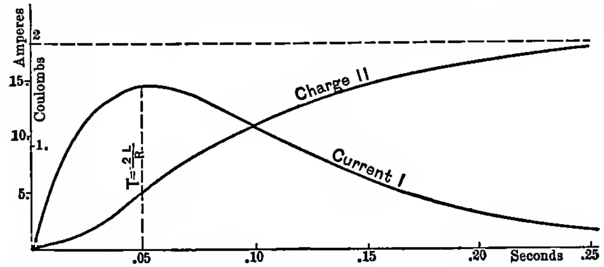

Fig. 28.—Just Non-oscillatory Charge of a Condenser with Capacity $C = 1000$ Microfarads through a Circuit with Resistance $R = 100$ Ohms, and Self-induction $L = 2.5$ Henrys.

a similar manner to curve II., Fig. 25, and, indeed, curve II. of Fig. 28 is identical with curve II. of Fig. 25, it being inverted and plotted downwards from the horizontal line.

CHAPTER IX.

CIRCUITS CONTAINING RESISTANCE, SELF-INDUCTION, AND CAPACITY.

CASE III. SOLUTION AND DISCUSSION FOR HARMONIC E. M. F.

CONTENTS :—To find from the general solutions the particular equations in the case of an harmonic E. M. F. Complete solutions for i and q. These same solutions obtained directly from the differential equations.

Discussion of Case III. Harmonic E. M. F.

The impediment. Case A. Circuits containing resistance and self-induction only. Case B. Circuits containing resistance and capacity only. Case C. Circuits containing resistance only. Case D. Circuits containing capacity only.

Effects of Varying the Constants of a Circuit.

First. Electromotive force varied. *Second.* Resistance varied. *Third.* Coefficient of self-induction varied. *Fourth.* Capacity varied. *Fifth.* The frequency varied.

The energy expended per second upon a circuit in which an harmonic current is flowing.

THE EQUATIONS FOR AN HARMONIC E. M. F. OBTAINED FROM THE GENERAL SOLUTION.

In the preceding cases considered, those of discharge and charge, the solutions for the value of the current and charge at any time were obtained in two ways, first from the general solution, and then directly from the differential equations, by substituting $e = f(t) = 0$, and $e = f(t) = E$, respectively.

The case of a circuit containing resistance, self-induction, and capacity, in which there is an impressed E. M. F. varying harmonically, will now be considered, and the solution derived first from the general equations (99) and (100), and then directly from the differential equations (89) and (90). In this case,

$$(174) \qquad e = f(t) = E \sin \omega t,$$

$$(175) \qquad \text{and} \qquad \frac{de}{dt} = f'(t) = E\omega \cos \omega t.$$

Substituting these values in (99) and (100), we have

$$(176) \quad i = \frac{CE\omega}{\sqrt{R^2C^2 - 4LC}} \left\{ \epsilon^{-\frac{t}{T_1}} \int \epsilon^{+\frac{t}{T_1}} \cos \omega t\, dt - \epsilon^{-\frac{t}{T_2}} \int \epsilon^{+\frac{t}{T_2}} \cos \omega t\, dt \right\} + c_1 \epsilon^{-\frac{t}{T_1}} + c_2 \epsilon^{-\frac{t}{T_2}},$$

and

$$(177) \quad q = \frac{CE}{\sqrt{R^2C^2 - 4LC}} \left\{ \epsilon^{-\frac{t}{T_1}} \int \epsilon^{+\frac{t}{T_1}} \sin \omega t\, dt - \epsilon^{-\frac{t}{T_2}} \int \epsilon^{+\frac{t}{T_2}} \sin \omega t\, dt \right\} + c_3 \epsilon^{-\frac{t}{T_1}} + c_4 \epsilon^{-\frac{t}{T_2}}.$$

The solution for q being similar to that for i, we will give the integration and reduction of (176) alone, and simply give the resulting expression for q. The integrals may be found by the formulæ of reduction [see equations (24) and (25), Chapter III.], obtained by integrating by parts. The integration of each term in (176) is

$$(178) \qquad \epsilon^{-\frac{t}{T}} \int \epsilon^{+\frac{t}{T}} \cos \omega t\, dt = \frac{1}{\frac{1}{T^2} + \omega^2} \left\{ \frac{1}{T} \cos \omega t + \omega \sin \omega t \right\}.$$

For convenience in transformation and reduction, put $\tau_1 = \frac{1}{T_1}$ and $\tau_2 = \frac{1}{T_2}$. After making these substitutions in equation (176), we have

$$(179)\quad i = \frac{CE\omega}{\sqrt{R^2C^2 - 4LC}} \left\{ \left(\frac{\tau_1}{\tau_1^2 + \omega^2} - \frac{\tau_2}{\tau_2^2 + \omega^2} \right) \cos \omega t \right.$$

$$\left. + \left(\frac{\omega}{\tau_1^2 + \omega^2} - \frac{\omega}{\tau_2^2 + \omega^2} \right) \sin \omega t \right\} + c_1 \epsilon^{-\frac{t}{T_1}} + c_2 \epsilon^{-\frac{t}{T_2}}.$$

We may simplify (179) by substituting the values of τ_1 and τ_2 [see (95)].

$$\tau_1 = \frac{1}{T_1} = \frac{RC - \sqrt{R^2C^2 - 4LC}}{2LC}.$$

$$\tau_2 = \frac{1}{T_2} = \frac{RC + \sqrt{R^2C^2 - 4LC}}{2LC}.$$

Then, after a few simple algebraic transformations in the coefficients of the sine and cosine, (179) becomes

$$(180)\quad i = \frac{ER\omega^2}{R^2\omega^2 + \left(\frac{1}{C} - L\omega^2\right)^2} \sin \omega t$$

$$+ \frac{E\omega\left(\frac{1}{C} - L\omega^2\right)}{R^2\omega^2 + \left(\frac{1}{C} - L\omega^2\right)^2} \cos \omega t + c_1 \epsilon^{-\frac{t}{T_1}} + c_2 \epsilon^{-\frac{t}{T_2}}.$$

This may be transformed into a more convenient form by means of the trigonometrical formula [see equation (27), Chapter III.]

$$A \sin x + B \cos x = \sqrt{A^2 + B^2} \sin \left(x + \tan^{-1} \frac{B}{A} \right),$$

and when transformed is written

$$(181)\quad i = \frac{E}{\sqrt{R^2 + \left(\frac{1}{C\omega} - L\omega\right)^2}}$$

$$\sin\left\{\omega t + \tan^{-1}\left(\frac{1}{CR\omega} - \frac{L\omega}{R}\right)\right\} + c_1 \epsilon^{-\frac{t}{T_1}} + c_2 \epsilon^{-\frac{t}{T_2}}.$$

This is the complete solution for the current in a circuit with resistance, self-induction, and capacity when the E. M. F. is harmonic and equal to $E \sin \omega t$. The discussion of this equation is deferred to the latter part of the chapter.

To Find the Equation for Charge.

The corresponding equation for charge, being the integral of the current according to the relation $q = \int i\,dt$, may be written

$$(182)\quad q = \frac{-E}{\omega\sqrt{R^2 + \left(\frac{1}{C\omega} - L\omega\right)^2}}$$

$$\cos\left\{\omega t + \tan^{-1}\left(\frac{1}{CR\omega} - \frac{L\omega}{R}\right)\right\} + c_3 \epsilon^{-\frac{t}{T_1}} + c_4 \epsilon^{-\frac{t}{T_2}}.$$

This equation is the complete solution for the charge in a circuit with resistance, self-induction, and capacity, when there is an harmonic impressed E. M. F.

To Obtain the Solution Directly from the Differential Equation.

Let us now proceed to obtain this same solution, equation (181), by solving the original differential equation, with the assumption that the E. M. F. varies harmonically,

that is, $e = E \sin \omega t$. Substituting $\frac{de}{dt} = E\omega \cos \omega t$ in the differential equation (89), we have

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = \frac{E\omega}{L} \cos \omega t. \tag{183}$$

This is a linear equation of the second order with constant coefficients. [See Johnson's Differential Equations, page 91]. The complete integral of such an equation consists of the sum of two parts, namely, the particular integral and the complementary function. The complementary function is the integral obtained by equating the first member to zero, and contains two arbitrary constants. The particular integral contains no arbitrary constants. The complementary function, obtained by equating the first member to zero and solving, is

$$i = c_1 \epsilon^{-\frac{t}{T_1}} + c_2 \epsilon^{-\frac{t}{T_2}}. \tag{101}$$

To find the particular integral, it is convenient to use the symbolic notation

$$D = \frac{d[\,]}{dt}. \qquad D^2 = \frac{d^2[\,]}{dt^2}.$$

With this notation (183) is written

$$\left(D^2 + \frac{R}{L} D + \frac{1}{LC}\right) i = \frac{E\omega}{L} \cos \omega t,$$

or

$$i = \frac{\frac{E\omega}{L} \cos \omega t}{\left(D^2 + \frac{R}{L} D + \frac{1}{LC}\right)}. \tag{184}$$

Next, to find the value of D^2, we have

$$\frac{d \cos \omega t}{dt} = D \cos \omega t = - \omega \sin \omega t,$$

$$\frac{d^2 \cos \omega t}{dt^2} = D^2 \cos \omega t = - \omega^2 \cos \omega t.$$

Therefore $\qquad D^2 = - \omega^2.$

Substituting in (184) $D^2 = - \omega^2$, we have

$$(185) \quad i = \frac{E \omega}{L\left(\frac{R}{L} D + \frac{1}{LC} - \omega^2\right)} \cos \omega t$$

$$= \frac{E \omega}{R D + \frac{1}{C} - L \omega^2} \cos \omega t.$$

Multiplying numerator and denominator of the coefficient of $\cos \omega t$ by $RD - \left(\frac{1}{C} - L \omega^2\right)$, we obtain

$$i = \frac{E \omega \left\{ RD - \left(\frac{1}{C} - L \omega^2\right) \right\}}{R^2 D^2 - \left(\frac{1}{C} - L \omega^2\right)^2} \cos \omega t.$$

Substituting $- \omega^2$ for D^2, and separating into two terms,

$$i = \frac{- E \omega R D \cos \omega t + E \omega \left(\frac{1}{C} - L \omega^2\right) \cos \omega t}{R^2 \omega^2 + \left(\frac{1}{C} - L \omega^2\right)^2}.$$

But $D \cos \omega t = - \omega \sin \omega t$. Hence

$$i = \frac{E \omega^2 R \sin \omega t}{R^2 \omega^2 + \left(\frac{1}{C} - L \omega^2\right)^2} + \frac{E \omega \left(\frac{1}{C} - L \omega\right) \cos \omega t}{R^2 \omega^2 + \left(\frac{1}{C} - L \omega^2\right)^2}$$

This is the particular integral, to which must be added the complementary function (101) in order to obtain the complete integral. The complete integral is thus found to be

$$(186) \quad i = \frac{E\,\omega^2 R \sin \omega t}{R^2\omega^2 + \left(\frac{1}{C} - L\,\omega^2\right)^2} + \frac{E\,\omega\left(\frac{1}{C} - L\,\omega^2\right)\cos \omega t}{R^2\omega^2 + \left(\frac{1}{C} - L\,\omega^2\right)^2} + c_1\,\epsilon^{-\frac{t}{T_1}} + c_2\,\epsilon^{-\frac{t}{T_2}}.$$

This solution for the current obtained from the differential equation (89) is seen to be identical with (180), the result obtained from the general solution (99). The solution for charge could be obtained in a similar manner from the differential equation (90).

Discussion of Case III.—Harmonic E. M. F.

These solutions, (181) and (182), show that, after a very short time has elapsed, so that the exponential terms containing the arbitrary constants of integration become inappreciably small and can be neglected, both the current and the charge are simple harmonic functions and may either lag behind or advance ahead of the impressed E. M. F. The current lags behind the impressed E. M. F., when $L\,\omega > \frac{1}{C\,\omega}$, and advances ahead of it when $L\,\omega < \frac{1}{C\,\omega}$. When $L\,\omega = \frac{1}{C\,\omega}$, that is, when $\omega = \frac{1}{\sqrt{LC}}$, there is no lag or advance, and the current is exactly in phase with the impressed E. M. F. In this case the current equation becomes

$$i = \frac{E}{R} \sin \omega t,$$

which is identical with the current equation obtained from Ohm's law, without considering either self-induction or

capacity. When the sine is unity in (181), the maximum value of the current, represented by I, is

$$(187) \qquad I = \frac{E}{\sqrt{R^2 + \left(\frac{1}{C\omega} - L\omega\right)^2}}.$$

From the analogy of this equation to Ohm's law, we see that the expression $\sqrt{R^2 + \left(\frac{1}{C\omega} - L\omega\right)^2}$ is of the nature of a resistance, and is the apparent resistance of a circuit containing resistance, self-induction, and capacity. This expression would quite properly be called "impedance," but the term impedance has for several years been used as a name for the expression $\sqrt{R^2 + L^2\omega^2}$, which is the apparent resistance of a circuit containing resistance and self-induction only [see equation (29), Chapter III.]. We would suggest, therefore, that the word "impediment" be adopted as a name for the expression $\sqrt{R^2 + \left(\frac{1}{C\omega} - L\omega\right)^2}$, which is the apparent resistance of a circuit containing resistance, self-induction, and capacity, and that the term impedance be retained in the more limited meaning it has come to have, that is, $\sqrt{R^2 + L^2\omega^2}$, the apparent resistance of a circuit containing resistance and self-induction only. Equation (187) may be written

$$(188) \qquad \text{Maximum current} = \frac{\text{Maximum E. M. F.}}{\text{Impediment}}.$$

Since the virtual current (the square root of the mean square of the instantaneous values of the current) is equal to $\frac{1}{\sqrt{2}}$ times the maximum value of the current, and since the virtual E. M. F. $= \frac{1}{\sqrt{2}}$ times the maximum E. M. F.,

$$(189) \qquad \text{Virtual current} = \frac{\text{Virtual E. M. F.}}{\text{Impediment}}.$$

It is convenient to consider the impediment as a resistance, and we are justified in so doing inasmuch as it has the same dimensions as a resistance, that is, a velocity in the electromagnetic system of units.

$$\omega = \frac{2\pi}{\text{Time}}.$$

$$L = \text{Length}.$$

Therefore, $$L\omega = \frac{\text{Length}}{\text{Time}} = \text{velocity}.$$

$$C = \frac{(\text{Time})^2}{\text{Length}}.$$

$$\frac{1}{C\omega} = \frac{\text{Length}}{\text{Time}} = \text{velocity}.$$

This gives the dimensions of a velocity to the whole expression for the impediment, which may therefore be considered as a resistance.

The several particular cases of circuits containing various combinations of resistance, self-induction, and capacity may readily be found by means of the general solution, equation (181).

Case A. Circuits containing Resistance and Self-induction only.

In this case the circuit has resistance R and self-induction L, and an harmonic E.M.F., $E \sin \omega t$. There being no condenser in the circuit, the capacity C is infinite [see page 67, Chapter IV.]. After the lapse of a very small time the terms containing the constants of integration in the general solution may be neglected as explained above. Substituting in (181) $C = \infty$, we have

$$i = \frac{E}{\sqrt{R^2 + L^2\omega^2}} \sin\left\{\omega t - \tan^{-1}\frac{L\omega}{R}\right\}.$$

This equation has been independently obtained from the differential equation [see equation (28), Chap. III.]. The current must always lag behind the impressed E. M. F. by an angle whose tangent is $\frac{L\omega}{R}$. In this case the impediment takes the particular value $\sqrt{R^2 + L^2\omega^2}$, which is known as the impedance of the circuit.

Case B. Circuits containing Resistance and Capacity only.

In this case the circuit has resistance R and capacity C, with an harmonic E. M. F., $e = E \sin \omega t$. Substituting $L = 0$ in the general equation (181), we have

$$i = \frac{E}{\sqrt{R^2 + \frac{1}{C^2\omega^2}}} \sin\left\{\omega t + \tan^{-1}\frac{1}{CR\omega}\right\}.$$

This equation has been independently obtained from the differential equation [see equation (78), Chapter V.]. The current must always advance ahead of the impressed E. M. F., when there is resistance and capacity only in the circuit, by an angle whose tangent is $\frac{1}{CR\omega}$.

Case C. Circuits containing Resistance Only.

In this case the self-induction $L = 0$, and the capacity $C = \infty$. Substituting these values in the general solution (181), we have

$$i = \frac{E}{R} \sin \omega t.$$

This result is immediately derivable from Ohm's law. Thus,

$$\text{Since} \quad e = E \sin \omega t,$$

$$\frac{e}{R} = \frac{E}{R} \sin \omega t,$$

$$\text{or} \qquad i = \frac{E}{R} \sin \omega t.$$

Case D. Circuits containing Capacity only.

In this case $R = 0$, and $L = 0$. Substituting in the general equation (181), we have

$$i = CE\omega \sin \left\{ \omega t + \frac{\pi}{2} \right\}.$$

This is identical with equation (80), Chapter V.

Effects of Varying the Constants of a Circuit.

The general equation (181) enables us to ascertain the current which will flow in a circuit when we know its resistance, self-induction, and capacity, the value of the impressed E. M. F. and its frequency. It is important to know two things about the current; first, its maximum value I, and, second, the angle θ by which it lags behind or advances ahead of the impressed E. M. F. The mean square value is readily obtained from the maximum value. We are given R, C, L, E, and ω. The angle of lag or advance is

$$\theta = \tan^{-1}\left(\frac{1}{CR\omega} - \frac{L\omega}{R}\right), \tag{190}$$

$$\text{or} \qquad \tan \theta = \frac{1}{CR\omega} - \frac{L\omega}{R}.$$

This is an angle of advance or lag, according as $\frac{1}{CR\omega}$ is

greater or less than $\frac{L\omega}{R}$. The maximum value for the current is

$$(191)\quad I = \frac{E}{\sqrt{R^2 + \left(\frac{1}{C\omega} - L\omega\right)^2}} = \frac{\frac{E}{R}}{\sqrt{1 + \tan^2\theta}} = \frac{E}{R}\cos\theta.$$

It is interesting to note how any change in R, L, C, ω, or E affects the value of θ and the current.

First. If the impressed E. M. F. E is varied, and R, L, C, and ω are maintained constant, θ is not affected, and the angle of lag or advance remains unchanged. The value of the current is varied in direct proportion to E.

$$I \propto E.$$

Second. If the resistance R of the circuit is varied, and L, C, ω, and E are maintained constant, as R is increased, the angle of lag or advance is diminished.

$$\tan\theta \propto \frac{1}{R}.$$

The sign of $\tan\theta$ is positive or negative, and the angle therefore one of advance or lag, according to the values of L, C, and ω, and is independent of any variations in the resistance. The current is in all cases diminished by an increase of resistance, but the amount of this decrease depends not only upon R, but upon the relation between $\frac{1}{C\omega}$ and $L\omega$.

In Fig. 29 are shown two particular cases of the variation in the current produced by change in the resistance. Curve I. is for a circuit in which

Self-induction $L = 2$ henrys $= 2 \times 10^9$ C. G. S. units.
Capacity $C = .55$ microfarads $= .55 \times 10^{-15}$ C. G. S. units.
Impressed E. M. F. $E = 200$ volts $= 200 \times 10^8$ C. G. S. units.
$2\pi n = \omega = 955.$

The abscissæ represent resistance in ohms (1 ohm $= 10^9$ C. G. S. units of resistance). The ordinates represent current in amperes (1 ampere $= 10^{-1}$ C. G. S. units). The

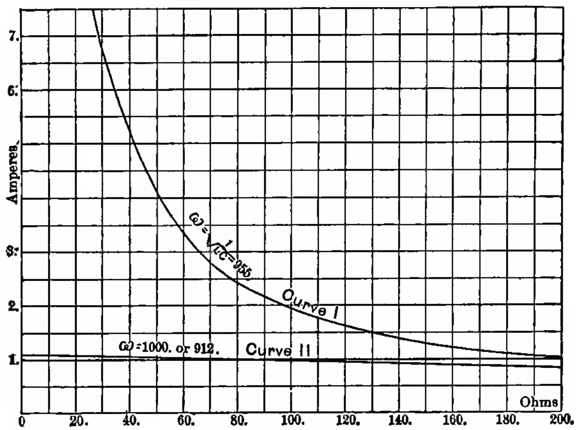

Fig. 29.—Variation of Current with Change in Resistance in a Circuit in which $E = 200$, $C = .55$, $L = 2$.

relation between L, C, and ω here taken is such that $\frac{1}{C\omega} = L\omega$, or $\omega = \frac{1}{\sqrt{LC}}$, which is the relation that gives no angle of lag or advance. The relation between current and resistance is the same as in Ohm's law, and when plotted gives the hyberbola curve I. In the same figure, curve II. represents the value of the current with different resistances in a circuit in which

$L = 2$ henrys, $E = 200$ volts,
$C = .55$ microfarads, $\omega =$ either 1000 or 912.

The constants here are the same as in the previous case, with the exception of ω, which has been changed from 955, that is, $\frac{1}{\sqrt{LC}}$, to either 1000 or 912. Any change in

ω from the value $\dfrac{1}{\sqrt{L\,C}}$, whether it be an increase or decrease, causes the curve to depart from the hyperbola, curve I. It is to be noticed that a change in frequency of only seven alternations per second will change the curve from I. to II.

Third. If the coefficient of self-induction L is varied while R, C, ω, and E are maintained constant,

When $L < \dfrac{1}{C\,\omega^2}$, $\tan\theta$ is positive and θ is an angle of advance.

$$\left.\begin{array}{l}\theta \text{ becomes less} \\ I \text{ becomes greater}\end{array}\right\} \text{ as } L \text{ increases.}$$

When $L > \dfrac{1}{C\,\omega^2}$, $\tan\theta$ is negative and θ is an angle of lag.

$$\left.\begin{array}{l}\theta \text{ becomes greater} \\ I \text{ becomes less}\end{array}\right\} \text{ as } L \text{ increases.}$$

These changes in the angle of lag or advance and the current, due to change in the self-induction, are better seen from the consideration of a particular case. In Fig. 30 the values of θ and I are plotted for various values of L in a circuit in which

$$R = 50 \text{ ohms}, \qquad \omega = 1000,$$
$$C = .55 \text{ microfarads}, \qquad E = 200 \text{ volts}.$$

When $L = \dfrac{1}{C\,\omega^2} = 1.82$, the current has its maximum value equal to $\dfrac{E}{R}$, and $\theta = 0$. This is a critical point, and a slight change of L in either direction will cause θ to reach a considerable value and the current to fall to a small part of the maximum value. If L be increased from 1.82 to 1.92, θ changes from zero to $-63°$, an angle of lag, and the current falls from 4 to 1.8 amperes. If L be made 1.72,

θ becomes an angle of advance of 63° and the current will be 1.8 amperes. It is thus seen that an exact balance of self-induction and capacity would be exceedingly hard to maintain in this case, for a slight change in the self-induction would cause a large angle of lag or advance and a large diminution in the current. Just how critical the curves will be in the vicinity of the point of equilibrium depends

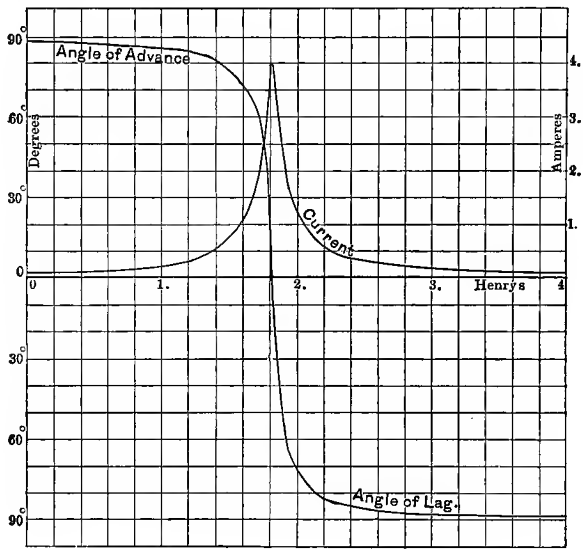

FIG 30.—VALUE OF CURRENT, AND ANGLE OF ADVANCE OR LAG FOR DIFFERENT AMOUNTS OF SELF-INDUCTION IN A CIRCUIT IN WHICH $R = 50$, $C = .55$, $E = 200$, $\omega = 1000$.

upon the constants of the circuit. The curves will always be of a form similar to those in Fig. 30, but will often be decidedly modified by the particular values of R, C, and ω. The critical parts of the curves may be more or less marked according to these particular values.

Fourth. If the capacity C is varied while R, L, ω, and E are maintained constant.

When $C < \frac{1}{L\omega^2}$, tan θ is positive, and θ is an angle of advance.

$$\left.\begin{array}{l}\theta \text{ becomes less} \\ I \text{ becomes greater}\end{array}\right\} \text{as } C \text{ increases.}$$

When $C > \frac{1}{L\omega^2}$, tan θ is negative and θ is an angle of lag.

$$\left.\begin{array}{l}\theta \text{ becomes greater} \\ I \text{ becomes less}\end{array}\right\} \text{as } C \text{ increases.}$$

These changes of current and lag, with the variation in capacity, are shown in Fig. 31 for a particular case in which

$R = 50$ ohms, $\omega = 1000$,
$L = 2$ henrys, $E = 200$ volts.

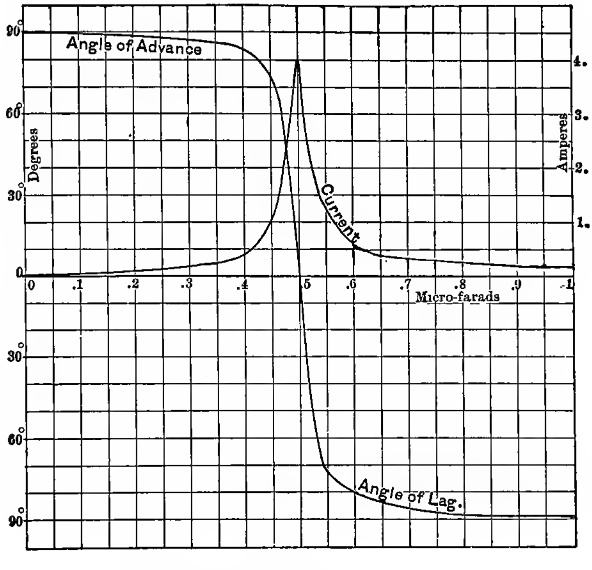

Fig. 31.—Value of Current, and Angle of Advance or Lag for Different Capacities in a Circuit in which $R = 50$, $L = 2$, $E = 200$, $\omega = 1000$.

The maximum value for the current occurs when $C = \frac{1}{L\omega^2} =$ .5 microfarads. This is a critical point in the curve similar to that in the curves where the self-induction was varied. Here θ is zero, and so the current, being in phase with the impressed E. M. F., has a value of 4 amperes in accordance with Ohm's law. The critical nature of the curves here is seen by the fact that when $C = .55$ there is an angle of lag of 75° and $I = 1.07$; when $C = .458$, there is an angle of advance of 75°. When C is changed from .5 to .488, the current falls from 4 to 2.83 amperes and is put 45° out of phase in advance of the E. M. F.

Fifth. If the frequency is varied while R, C, L, and E are maintained constant, still more marked changes occur in the values of I and θ.

When $\omega < \frac{1}{\sqrt{LC}}$, tan θ is positive and θ is an angle of advance.

$$\left.\begin{array}{l}\theta \text{ becomes less} \\ I \text{ becomes greater}\end{array}\right\} \text{as } \omega \text{ increases.}$$

When $\omega > \frac{1}{\sqrt{LC}}$, tan θ is negative and θ is an angle of lag.

$$\left.\begin{array}{l}\theta \text{ becomes greater} \\ I \text{ becomes less}\end{array}\right\} \text{as } \omega \text{ increases.}$$

In Fig. 32 the values of the current and angle of lag are shown for different values of ω in a circuit in which

$$R = 50 \text{ ohms}, \qquad C = .55 \text{ microfarads},$$
$$L = 2 \text{ henrys}, \qquad E = 200 \text{ volts}.$$

When $\omega = \frac{1}{\sqrt{LC}} = 955$, the current has its maximum value of 4 amperes, in accordance with Ohm's law. Here $\theta = 0$. A change of five per cent, one way or the other in this critical value for ω causes an angle of lag or advance

of 75°, and the current falls to one-fourth of the maximum. Just how critical the curves are, in the vicinity of this point

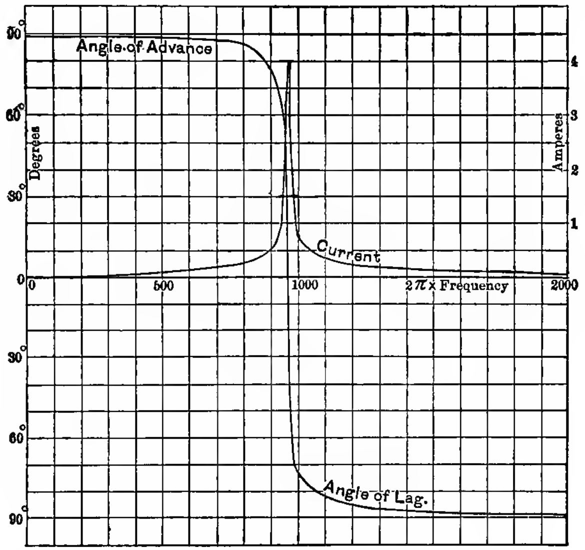

FIG. 32.—VALUE OF CURRENT AND ANGLE OF ADVANCE OR LAG FOR DIFFERENT FREQUENCIES IN A CIRCUIT IN WHICH $R = 50$, $L = 2$, $C = .55$, $E = 200$.

of equilibrium depends upon the particular values of R, C, and L.

In Fig. 33 is shown the E. M. F. necessary to cause a constant current to flow in a circuit in which R, C, and ω are constant. In the particular case plotted,

$$R = 50 \text{ ohms}, \qquad I = 1 \text{ ampere},$$
$$C = .55 \text{ microfarads}, \qquad \omega = 1000.$$

As the self-induction is increased up to the value $L = \frac{1}{C\omega^2} = 1.82$, the E. M. F. needed to drive the current becomes less and less, and when $L = 1.82$ the E. M. F. needed is only 50 volts. As L increases past this critical value, the value of the E. M. F. needed increases. Except

very near the critical point, the change in the necessary E. M. F. is almost directly proportional to the change in the self-induction, that is, the curve is formed of two straight lines with a rounded point. This curve is the reciprocal of

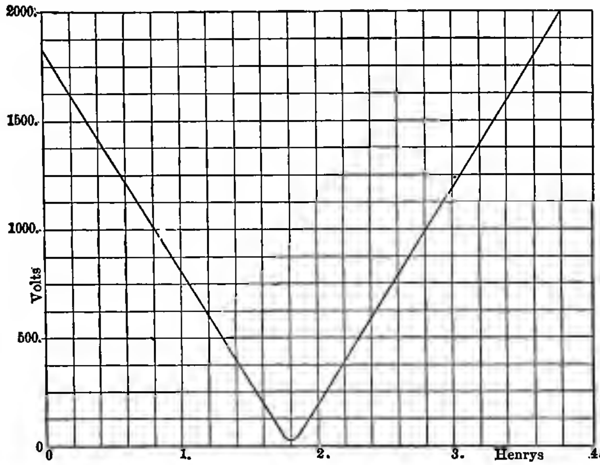

FIG. 33.—RELATION BETWEEN IMPRESSED E. M. F. AND SELF-INDUCTION WHEN 1 AMPERE FLOWS IN A CIRCUIT IN WHICH $R = 50$, $C = .55$, $\omega = 1000$.

the corresponding curve for current, with E constant and L variable, as shown in Fig. 30.

THE ENERGY EXPENDED PER SECOND UPON A CIRCUIT IN WHICH AN HARMONIC CURRENT IS FLOWING.

The energy expended in any circuit in the time dt is the product of the E. M. F. and current at that instant by the time; that is,

$$dW = ei\,dt. \quad \text{[See equation (5), Chap. I.]}$$

When the E. M. F. is harmonic the instantaneous value of it is $e = E \sin \omega t$. The current at the same instant is $i = I \sin \{\omega t - \theta\}$. Therefore the differential equation of energy is

$$(192) \qquad dW = EI \sin \omega t \sin \{\omega t - \theta\}\, dt.$$

Integrating between the limits zero and T, the time of one complete period, we obtain

(193) $$W = EI\int_0^T \sin \omega t \sin \{\omega t - \theta\} d t.$$

Expanding $\sin \{\omega t - \theta\}$, we obtain

$$W = EI \cos \theta \int_0^T \sin^2 \omega t \, d t - EI \sin \theta \int_0^T \sin \omega t \cos \omega t.$$

Replacing $\sin \omega t \cos \omega t$ by its equivalent $\frac{\sin 2 \omega t}{2}$,

$$W = EI \cos \theta \int_0^T \sin^2 \omega t \, d t - \frac{EI \sin \theta}{2} \int_0^T \sin 2 \omega t \, d t.$$

Between the limits zero and T the second integral vanishes and the first integral is equal to $\frac{T}{2}$. [See page 37.] The value of the energy expended per period is therefore

(194) $$W = \frac{EI \cos \theta}{2} T.$$

The energy expended per second is therefore

(195) $$W = \frac{EI \cos \theta}{2};$$

that is, the energy per second is half the product of the maximum E. M. F. by the maximum current by the cosine of the angle of difference between the E. M. F. and current. Since the effective E. M. F. or current is equal to $\frac{1}{\sqrt{2}}$ times the maximum value, we have

(196) $$W = \bar{E}\bar{I} \cos \theta,$$

meaning by $\bar{E}$ and $\bar{I}$ the square root of the mean square values of E. M. F. and current.

CHAPTER X.

CIRCUITS CONTAINING RESISTANCE, SELF INDUCTION, AND CAPACITY.

CASE III. (CONTINUED.) CURRENTS AT THE "MAKE" FOR AN HARMONIC E. M. F.

CONTENTS:—Complete equations for i and q with the complementary function in the oscillatory form. To determine the constants A' and Φ'. To determine the constants A and Φ. Complete equation for i with constants determined. Examples to explain the general equation in cases of particular circuits. Curves showing the current at the make for a particular circuit. The phase at which the E. M. F. should be introduced to make the oscillation a maximum.

IN the discussion of the current equation in Chapter IX. for an harmonic E. M. F., it was stated that after the lapse of a very short time the exponential terms, equation (181), become inappreciably small and can be neglected, and the discussion of the equation there given only applies after the current has been flowing for a short time. It is proposed in this chapter to investigate the effect of these exponential terms in modifying the current during the very short time after the "make," or, in other words, after the harmonic E. M. F. is suddenly introduced into the circuit. The E. M. F. may be introduced at any point of its phase, that is, it may be zero or may have its maximum or any intermediate value, but, in any case, the complete equations (181) and (182) show just what happens, provided we determine the constants c_1 and c_2 of the complementary func-

tion, so that they correspond to the particular hypothesis made.

It has been noted (120) that the complementary function $c_1\,\epsilon^{-\frac{t}{T_1}} + c_2\,\epsilon^{-\frac{t}{T_2}}$ may be written in another form, viz.:

$$A\,\epsilon^{-\frac{Rt}{2L}} \sin\left\{\frac{\sqrt{4\,L\,C - R^2\,C^2}}{2\,LC}\,t + \Phi\right\}.$$

This latter form must be used when we have the relation $4\,L > R^2 C$, for, under this hypothesis, the time-constants T_1 and T_2 of the first form become imaginary. To make this supposition is equivalent to saying that the character of the discharge from the circuit is oscillatory [see Chapter VII.]. Inasmuch as this relation $4\,L > R^2 C$ is true for most ordinary circuits in which L has an appreciable value, and since the results obtained are rather more interesting under this supposition than under the supposition that $4\,L < R^2 C$, which would give "dead beat" discharge, we will confine our attention to the oscillatory case only. The plan to be followed in the discussion of this subject will be to determine the constants A and Φ of the general equation, and write the general result. The application of this result to a particular circuit will then be made, and curves drawn showing the current as it starts in this circuit before it has reached its final harmonic form.

The general equation for current, under the assumption made that $4\,L > R^2 C$, may be written

$$(197)\quad i = \frac{E}{\sqrt{R^2 + \left(\frac{1}{C\,\omega} - L\,\omega\right)^2}} \sin\left\{\omega\,t + \tan^{-1}\left(\frac{1}{C\,R\,\omega} - \frac{L\,\omega}{R}\right)\right\} + A\,\epsilon^{-\frac{Rt}{2L}} \sin\left\{\frac{\sqrt{4\,L\,C - R^2\,C^2}}{2\,L\,C}\,t + \Phi\right\}.$$

where A and Φ are the constants of integration to be determined and are each of them real. Likewise, the equation expressing the quantity of charge on the condenser at any moment may be written [see (182) and (123)]

$$(198)\quad q = \frac{-E}{\omega\sqrt{R^2+\left(\frac{1}{C\omega}-L\omega\right)^2}}\cos\left\{\omega t+\tan^{-1}\left(\frac{1}{CR\omega}-\frac{L\omega}{R}\right)\right\}+A'\epsilon^{-\frac{Rt}{2L}}\sin\left\{\frac{\sqrt{4LC-R^2C^2}}{2LC}t+\Phi'\right\}.$$

To determine the constants A' and Φ';—Remembering the relation $dq = i\,dt$, we may differentiate (198) and write

$$(199)\quad i=\frac{dq}{dt}=\frac{E}{\sqrt{R^2+\left(\frac{1}{C\omega}-L\omega\right)^2}}\sin\left\{\omega t+\tan^{-1}\left(\frac{1}{CR\omega}-\frac{L\omega}{R}\right)\right\}+\frac{A'}{\sqrt{LC}}\epsilon^{-\frac{Rt}{2L}}\sin\left\{\frac{\sqrt{4LC-R^2C^2}}{2LC}t+\Phi'-\tan^{-1}\frac{\sqrt{4LC-R^2C^2}}{RC}\right\}.$$

Equating (199) with (197), we obtain the relations

$$(200)\quad A=\frac{A'}{\sqrt{LC}},$$

$$(201)\quad \Phi=\Phi'-\tan^{-1}\frac{\sqrt{4LC-R^2C^2}}{RC}.$$

For simplification make the following substitutions:

$$(202)\quad I=\frac{E}{\sqrt{R^2+\left(\frac{1}{C\omega}-L\omega\right)^2}}.\qquad \text{[See (191).]}$$

(203) $$\psi = \omega t + \tan^{-1}\left(\frac{1}{C R \omega} - \frac{L \omega}{R}\right) = \omega t + \theta.$$

(204) $$\alpha = \frac{\sqrt{4 L C - R^2 C^2}}{2 L C}.$$

The frequency of oscillation is $\frac{\alpha}{2\pi}$, and the period $\frac{2\pi}{\alpha}$. Then we may write, after substituting in (197) and (198) the values of A' and Φ' as determined,

(205) $$i = I \sin \psi + A \epsilon^{-\frac{R t}{2 L}} \sin \{\alpha t + \Phi\}.$$

(206) $$q = -\frac{I}{\omega} \cos \psi + A \sqrt{L C} \epsilon^{-\frac{R t}{2 L}}$$

$$\sin \left\{ \alpha t + \Phi + \tan^{-1} \frac{\sqrt{4 L C - R^2 C^2}}{R C} \right\}.$$

To determine the constants A and Φ:—In these equations time is counted from the point when the impressed E. M. F. is zero. Let t_1 be the time when the E. M. F. is introduced. We know then that the current and the charge of the condenser are each zero at the time t_1, the condenser having no initial charge. These conditions alone, namely, that $i = 0$ and $q = 0$ when $t = t_1$, are sufficient to determine the constants. In equations (205) and (206) make $i = 0$, $q = 0$ when $t = t_1$, and call ψ_1 the value of ψ when $t = t_1$, and we have

(207) $$0 = I \sin \psi_1 + A \epsilon^{-\frac{R t_1}{2 L}} \sin \{\alpha t_1 + \Phi\}.$$

(208) $$0 = -\frac{I}{\omega} \cos \psi_1 + A \sqrt{L C} \epsilon^{-\frac{R t_1}{2 L}}$$

$$\sin \left\{ \alpha t_1 + \Phi + \tan^{-1} \frac{\sqrt{4 L C - R^2 C^2}}{R C} \right\}.$$

Eliminating A between these equations, we obtain

$$(209) \qquad \Phi = \cot^{-1} - \left\{ \frac{2 \cot \psi_1 + R\, C\, \omega}{\omega \sqrt{4\, L\, C - R^2 C^2}} \right\} - \alpha\, t_1 .$$

Substituting this value of Φ in (207),

$$(210) \qquad A = - I \epsilon^{+\frac{R t_1}{2L}} \frac{\sin \psi_1}{\sin \cot^{-1} - \left\{ \frac{2 \cot \psi + R\, C\, \omega}{\omega \sqrt{4\, L\, C - R^2\, C^2}} \right\}} .$$

This expression for A may be reduced by simple trigonometrical operations to the form

$$(211) \quad A = - \frac{2 I \epsilon^{+\frac{R t_1}{2L}}}{\omega \sqrt{4\, L\, C - R^2 C^2}}$$

$$\sqrt{(L\, C\, \omega^2 - 1) \sin^2 \psi_1 + \tfrac{1}{2} R\, C\, \omega \sin 2\, \psi_1 + 1}.$$

Substituting these values of A and Φ in equation (197), we may write the complete solution with constants determined,

$$(212)\; i = I \sin \psi - \frac{2\, I \sqrt{(L C \omega^2 - 1) \sin^2 \psi_1 + \frac{1}{2} R C \omega \sin 2 \psi_1 + 1}}{\omega \sqrt{4\, L\, C - R^2 C^2}}$$

$$\epsilon^{-\frac{R}{2L}(t - t_1)} \sin \left\{ \alpha\, (t - t_1) + \cot^{-1} - \left[\frac{2 \cot \psi_1 + C R\, \omega}{\omega \sqrt{4 L C - R^2 C^2}} \right] \right\} .$$

There are several general conclusions which can be made in interpreting the meaning of this equation. It is evident that there will be an oscillation of the current when the E. M. F. is first introduced, which gradually dies away, the rate of dying away depending upon the exponent of ϵ in the equation or, in other words, upon the time-constant of the circuit, namely, $\frac{2\, L}{R}$. The initial value of this logarithmic decrement curve, that is at the make when $t = t_1$, is expressed by the coefficient of ϵ in the equation. It is evident

that this initial value depends upon the value of ψ_1 for its value, or is a function of ψ_1. The initial value of the logarithmic curve has, then, a different value for every value of ψ_1, i.e., for every point of the phase of what the current would have been if it had started out at the make as an harmonic current having the same phase difference with the E. M. F. as it finally assumes. Again, at the time $t = t_1$ the value of the last term of the equation becomes $-I \sin \psi_1$. This will be evident upon replacing the coefficient of ϵ by its value given in (210). The first term becomes $I \sin \psi_1$, when $t = t_1$, and the two terms together show that the equation makes the value of the current zero at the time t_1, that is, at the time the E. M. F. is introduced.

In order to show the meaning of this equation more clearly, a particular example will be assumed. Suppose we have a circuit with a resistance of 50 ohms, a self-induction of 2 henrys, and a capacity of .55 microfarads, all in series. Such a circuit would correspond nearly to the fine wire coil of a small 10-light Westinghouse transformer connected in series with a condenser of .55 microfarads capacity. Let an E. M. F. of 100 volts (maximum value), having a periodicity of 159, be impressed upon the circuit; that is, the angular velocity $\omega = 2\pi \times 159 = 1000$, approximately. We have, then, with these values assumed,

$$
\begin{aligned}
E &= 100 \text{ volts (max.)} = 100 \times 10^8 \text{ C. G. S. units.}\\
R &= 50 \text{ ohms} \quad = 50 \times 10^9 \text{ C. G. S. units.}\\
L &= 2 \text{ henrys} \quad = 2 \times 10^9 \text{ C. G. S. units.}\\
C &= .55 \text{ microfarads} = .55 \times 10^{-15} \text{ C. G. S. units.}\\
\omega &= 1000.\\
T &= \frac{2L}{R} = \frac{4 \times 10^9}{50 \times 10^9} = .08 \text{ seconds.}\\
I &= .53 \text{ amperes (max.).} \quad \text{[See equation (202).]}\\
\theta &= -74^\circ\, 30'.\\
\alpha &= 955 = 2\pi \times \text{frequency of oscillation} = 2\pi \times 151.\\
&\quad \text{[See (204).]}
\end{aligned}
$$

The equation for the current in this particular case is

$$(213)\quad i = .53 \sin \psi - .477 \sqrt{.1 \sin^2 \psi_1 + .0137 \sin 2\psi_1 + 1}\; \epsilon^{-\frac{t-t_1}{.08}} \sin \{955 (t - t_1) + \chi\}.$$

Curve III., Fig. 34, represents the plot of this equation when the particular value of ψ_1 is 30°. This means that the E. M. F. is introduced into the circuit at that particular time at which the normal current curve is 30° from its zero value. The value of the coefficient of ϵ when ψ_1 is 30° is .495, and the equation reads

$$(214)\quad i = .53 \sin \psi - .495\, \epsilon^{-\frac{t-t_1}{.08}} \sin \{955 (t - t_1) + \chi\}.$$

Here χ stands for angle of lag expressed in equation (212), and is not expressed in figures inasmuch as it is not necessary to know it in order to draw the curves, because the phase is determined by the fact that we know the distance $O' A'$, Fig. 34, it being equal but of opposite sign to the distance $O A$. It will be noticed that the initial value of the logarithmic decrement is nearly the same for any value of ψ_1 in this particular case. Moreover, as it happens, the initial value of the logarithmic decrement is nearly the same as the maximum value of the current I. Curve I. is a sine curve representing the first term in equation (214), and curve II. a sine curve with logarithmic decrement representing the second term in the equation. The current curve, III., is the sum of curves I. and II. After about one-tenth of a second, curve II. becomes inappreciable and the current follows a simple sine curve.

As a second example, let us consider the same circuit as before. But now suppose the frequency is just half what it was in the first example, namely, 79.5, or that $\omega = 500$. Furthermore, suppose the E. M. F. is such that it will send a maximum current of .5 of an ampere through

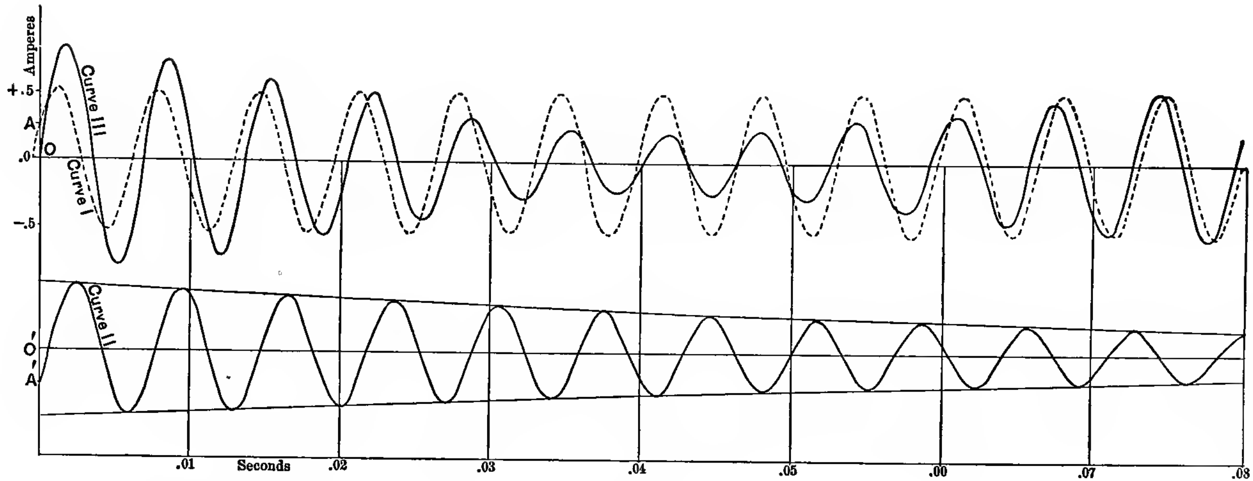

FIG. 34.—CURVE III. SHOWS THE CURRENT WHICH FLOWS AFTER THE INTRODUCTION OF AN HARMONIC E. M. F. INTO A CIRCUIT WITH R, L, AND C. IT IS THE SUM OF THE TWO COMPONENT CURVES, I. A SINE-CURVE, AND II. A SINE-CURVE WITH AN AMPLITUDE DECREASING ACCORDING TO A LOGARITHMIC DECREMENT.

the circuit. It will be found, upon calculation, that the E. M. F. must be 1320 volts maximum. With these values, then,

$$\begin{aligned} E &= 1320, \\ R &= 50, \\ L &= 2, \\ C &= .55, \\ \omega &= 500, \\ T &= .08 \text{ seconds}, \\ I &= .5 \text{ amperes}, \\ \theta &= 88^\circ\ 55', \qquad \tan\theta = 52.8, \\ \alpha &= 955, \end{aligned}$$

the equation for the current becomes

$$(215) \quad i = .5 \sin\psi - .955 \sqrt{-.725 \sin^2\psi_1 + .0069 \sin 2\psi_1 + 1}\; \epsilon^{-\frac{t-t_1}{.08}} \sin\{955(t-t_1)+\chi\}.$$

The plot of this equation, when ψ_1 is taken equal to 180° (that is, the E. M. F. is introduced when the normal current curve is zero), is shown in Fig. 35. It will be noticed that the initial value of the logarithmic curve has considerable variation according to the particular point of time at which the E. M. F. is introduced. This variation is represented in the curve IV., Fig. 35. The initial value of the logarithmic decrement at 0° or 180° is almost twice as much as the maximum value of the current I, their ratio being $\frac{.955}{.5}$. The equation, when ψ_1 is 180°, reduces to

$$(216) \quad i = .5 \sin\psi - .955\, \epsilon^{-\frac{(t-t_1)}{.08}} \sin\{955(t-t_1)+\chi\}.$$

In each of the above examples the current follows the sine law in about one-quarter of a second after the periodic E. M. F. is introduced, during which time somewhere in the neighborhood of forty oscillations have been made.

The phase at which the E. M. F. should be introduced to make the oscillation a maximum:—It may be interesting to inquire at what point the E. M. F. should be introduced

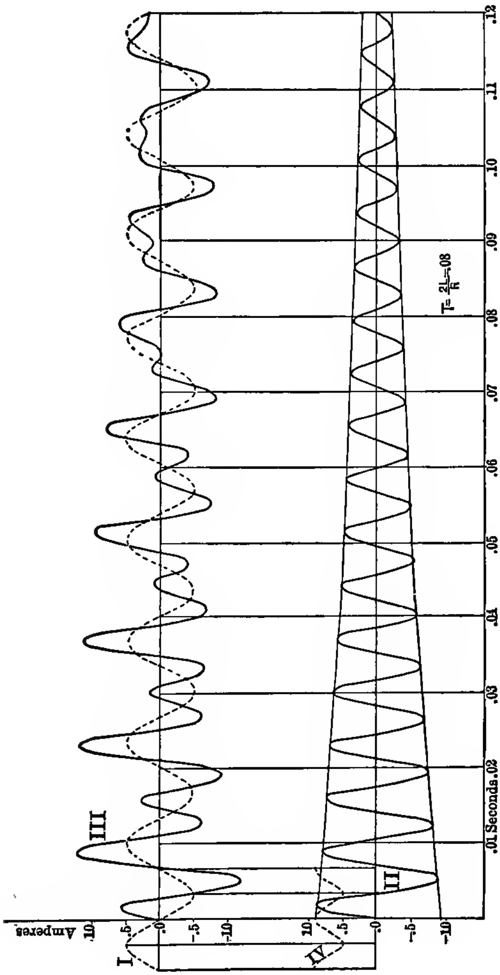

FIG. 35.—CURVE III. SHOWS THE CURRENT WHICH FLOWS AFTER THE INTRODUCTION OF HARMONIC E. M. F. INTO A CIRCUIT WITH *R*, *L*, AND *C*. IT IS THE SUM OF THE TWO COMPONENT CURVES, I. A SINE-CURVE, AND II. A SINE-CURVE WITH AN AMPLITUDE DECREASING ACCORDING TO A LOGARITHMIC DECREMENT.

into the circuit to render the effect of the oscillation a maximum. This point may readily be found by referring to equation (212). The coefficient of ϵ becomes a maximum

(for a variation in t_1), when the quantity under the radical sign is a maximum. Differentiating the quantity under the

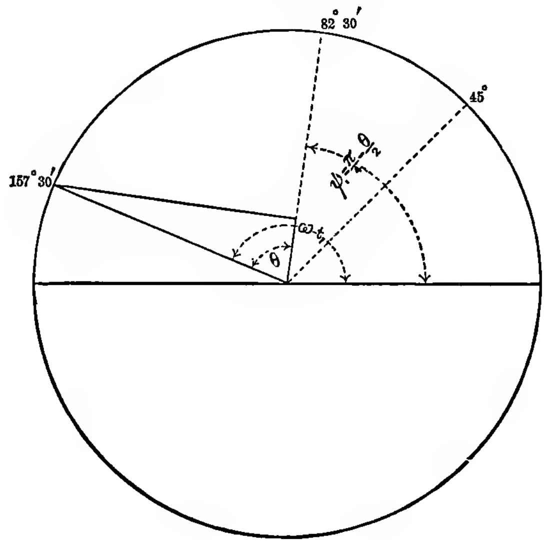

FIG. 36.—SHOWING HOW TO FIND GEOMETRICALLY THE ANGLE ψ_1 WHICH MAKES THE EFFECT OF THE EXPONENTIAL TERM A MAXIMUM.

radical, then, with respect to t_1, and equating to zero, we obtain

$$(217) \qquad (L\,C\,\omega^2 - 1)\sin 2\,\psi_1 + R\,C\,\omega\cos 2\,\psi_1 = 0.$$

$$\text{Whence} \quad \tan 2\,\psi_1 = \frac{R\,C\,\omega}{1 - L\,C\,\omega^2}.$$

But it will be remembered that [see equation (190)]

$$\tan\theta = \frac{1 - L\,C\,\omega^2}{R\,C\,\omega}.$$

$$\text{Hence} \quad \tan 2\psi_1 = \cot \theta = \tan\left(\frac{\pi}{2} - \theta\right),$$

(218) $$\text{or} \quad \psi_1 = \frac{\pi}{4} - \frac{\theta}{2}.$$

And since $\psi_1 = \omega t_1 + \theta$ [see (138)], we find

(219) $$\omega t_1 = \frac{\pi}{4} - \frac{3\,\theta}{2}.$$

Suppose θ is an angle of lag of $-\,75°$, as in the first example cited, then its sign is negative and $\psi_1 = \frac{\pi}{4} + \frac{75°}{2} = 82°\,30'$ for a maximum. If θ is $+\,88°\,55'$, as in the second example, $\psi_1 = 45° - 44°\,27'.5 = 32'.5$ for a maximum.

The curve IV., Fig. 35, shows that the maximum point is nearly at the position where $\psi = 0$, and thus agrees with this result. The exact form which the current curve assumes at the introduction of an harmonic E. M. F. depends upon the time of its introduction and the constants of the circuit. The curves shown in Figs. 34 and 35 give an idea of what may be expected in other cases. In all cases, after a very few periods, the current reaches the simple sine form.

The current which flows upon making a circuit which contains resistance and self-induction, but no capacity, is shown in Fig. 15, Chapter III., to which the reader is referred.

CHAPTER XI.

CIRCUITS CONTAINING RESISTANCE, SELF-INDUCTION, AND CAPACITY.

CASE IV. ANY PERIODIC E. M. F.

CONTENTS:—Fourier's theorem. General equations for i and q with any periodic E. M. F. If the self-induction and capacity neutralize each other at every point of time and the current is therefore the same as if both self-induction and capacity were absent, the impressed E. M. F. must be a simple harmonic E. M. F. If the heating effect, or any effect which depends upon $\int i^2 dt$, in a circuit, is the same when the self-induction and capacity are present as it is when they are absent, the impressed E. M. F. must be a simple harmonic E. M. F. Various types of current curves. When curves are not symmetrical, although the quantity flowing in the positive direction is equal to the quantity in the negative direction, yet the $\int i^2 dt$ effect will generally be different in these two directions. Illustration from a particular curve. Alternating-current arc-light carbons.

IF we suppose that the impressed E. M. F. is made up of a number of simple harmonic E. M. F.'s added together, the impressed E. M. F. may be written

$$(220) \quad e = E_1 \sin(b_1 \omega t + \theta_1) + E_2 \sin(b_2 \omega t + \theta_2) + E_3 \sin(b_3 \omega t + \theta_3) + \text{etc.}$$

and, therefore,

$$\frac{de}{dt} = E_1 b_1 \omega \cos(b_1 \omega t + \theta_1) + E_2 b_2 \omega \cos(b_2 \omega t + \theta_2) + \text{etc.}$$

Expressed as a summation, we have

$$(221) \qquad e = \sum_{E,b,\theta} E \sin(b\,\omega\,t + \theta) = f(t).$$

$$(222) \qquad \frac{de}{dt} = \omega \sum_{E,b,\theta} E b \cos(b\,\omega\,t + \theta) = f'(t).$$

In this summation it is to be understood that E and θ take in succession any values, fractional or integral, but that b may only have positive integral values as the E. M. F. is supposed to be periodic, and consequently the periods of the component sine-curves must be commensurable. It was shown by Fourier, in his treatise on the Analytical Theory of Heat, published in 1822, that such an expression as (220) or (221) represents any single-valued periodic function whatever, and is therefore an expression which represents any possible E. M. F. whatever. If (222) is substituted in the general equation for current (99), and (221) in the general equation for charge (100), it will be found, upon integrating, that each component term in the E. M. F. gives a term in the current or charge similar to that given in equations (181) and (182) in Case III., and consequently the resultant current may be expressed as a summation thus:

$$(223) \quad i = \sum_{E,b,\theta} \frac{E}{\sqrt{R^2 + \left(\frac{1}{C b \omega} - L b \omega\right)^2}} \sin \left\{ b\,\omega\,t + \theta + \tan^{-1}\left(\frac{1}{C R b \omega} - \frac{L b \omega}{R}\right) \right\} + c_1 \epsilon^{-\frac{t}{T_1}} + c_2 \epsilon^{-\frac{t}{T_2}},$$

and the charge

$$(224) \quad q = \sum_{E,b,\theta} \frac{E}{b\,\omega \sqrt{R^2 + \left(\frac{1}{C b \omega} - L b \omega\right)^2}} \cos \left\{ b\,\omega\,t + \theta + \tan^{-1}\left(\frac{1}{C R b \omega} - \frac{L b \omega}{R}\right) \right\} + c_3 \epsilon^{-\frac{t}{T_1}} + c_4 \epsilon^{-\frac{t}{T_2}}.$$

In these sums for i and q there must be as many terms in each as there are in the expression for the E. M. F., and the values of E, b, and θ must be the same in corresponding terms. These equations express the current and charge in a circuit whose E. M. F. is any periodic E. M. F., as in equation (221).

If the self-induction and capacity neutralize each other at every point of time, and the current is therefore the same as if both self-induction and capacity were absent, the impressed E. M. F. must be a simple harmonic E. M. F.—In the discussion of Case III., where the E. M. F. was harmonic and the resulting current was shown to be harmonic also, it was pointed out that if the relation $\omega = \frac{1}{\sqrt{L\,C}}$ existed, the current was the same as if there was no self-induction and no condenser in the circuit, and the same as if it simply followed Ohm's law. This was shown by substituting the relation $\omega = \frac{1}{\sqrt{L\,C}}$, or $\frac{1}{C\,\omega} - L\,\omega = 0$, in the current or charge equations (181) and (182) and neglecting the complementary function. Those equations, with these substitutions, become

$$i = \frac{E}{R}\sin\omega\,t.$$

$$q = -\frac{E}{R\,\omega}\cos\omega\,t.$$

It is seen that the current and charge are the same at every point of time as if the self-induction and capacity were absent. Now, since the current is the same at every point of time, the effects of this current will be the same; namely, the quantity which flows in a half period, being $\int_0^{\frac{T}{2}} i\,dt = Q$, is the same as when there is no self-induction

and capacity, and the energy expended in the circuit in performing work, or in heating effects, is likewise the same, being proportional to $\int i^2 dt$.

In order to ascertain whether some similar relation between self-induction and capacity would cause them to neutralize each other when the impressed E. M. F. is not a simple harmonic function of the time, consider the case where the E. M. F. is composed of two parts, each a sine-function of the time. Suppose

$$e = E_1 \sin a\,\omega t + E_2 \sin b\,\omega t, \tag{225}$$

where a and b are integers. In the circuit there is resistance, self-induction, and capacity. Then at any time the value of the current is [see (223)]

$$i = \frac{E_1}{\sqrt{R^2 + \left(L a\,\omega - \dfrac{1}{C a\,\omega}\right)^2}} \sin\left\{a\,\omega t + \tan^{-1}\frac{1}{R}\left(\frac{1}{C a\,\omega} - L a\,\omega\right)\right\}$$
$$+ \frac{E_2}{\sqrt{R^2 + \left(L b\,\omega - \dfrac{1}{C b\,\omega}\right)^2}} \sin\left\{b\,\omega t + \tan^{-1}\frac{1}{R}\left(\frac{1}{C b\,\omega} - L b\,\omega\right)\right\}. \tag{226}$$

Suppose the self-induction and capacity have the relation $a\,\omega = \dfrac{1}{\sqrt{L\,C}}$. Then they will neutralize each other in the first term of the above expression for the instantaneous value of the current. But in the second term the relation $b\,\omega = \dfrac{1}{\sqrt{L\,C}}$ is necessary to cause the self-induction and

capacity to neutralize each other. Now, if one of the above terms is changed by the introduction of self-induction and capacity, while the other term is unaffected, the value of the current which is equal to the sum of the two terms must be changed. It therefore follows that neither the relation $a\,\omega = \frac{1}{\sqrt{L\,C}}$ nor $b\,\omega = \frac{1}{\sqrt{L\,C}}$ will cause the self-induction and capacity to neutralize each other when introduced into a circuit containing an impressed E. M. F. composed of two simple harmonic E. M. F.'s with angular velocities $a\,\omega$ and $b\,\omega$, respectively. If $a = b$, the two terms in the expression for the instantaneous value of the current may be written as one, and we have a simple harmonic function of the time. The relation $a\,\omega = b\,\omega = \frac{1}{\sqrt{L\,C}}$ will then cause the self-induction and capacity to neutralize each other.

If $E_1 = 0$, or if $E_2 = 0$, then we have a simple sine-function, and the relation $b\,\omega = \frac{1}{\sqrt{L\,C}}$, or $a\,\omega = \frac{1}{\sqrt{L\,C}}$, respectively, will cause the balancing of the self-induction and capacity.

In order to ascertain the conditions under which there may be self-induction and capacity in a circuit, just neutralizing each other, so that the instantaneous values of the current will be the same as though there were no self-induction and capacity in the circuit, we will consider the general differential equation of E. M. F.'s

$$e = R\,i + L\frac{d\,i}{d\,t} + \frac{\int i\,d\,t}{C}.$$

[See equation (87).] We wish to ascertain the conditions by which the current will be the same as when there is neither self-induction nor capacity, that is, the conditions

by which $i = \frac{e}{R}$ and $e = Ri$, according to Ohm's law. Substituting in the above equation, we have

$$L\frac{di}{dt} + \frac{\int i\,dt}{C} = 0. \tag{227}$$

This is the same as saying that the E. M. F.'s of self-induction and capacity are equal and opposite. By differentiation,

$$\frac{d^2 i}{dt} = -\frac{i\,dt}{LC}.$$

Multiplying by $\frac{di}{dt}$,

$$\left(\frac{di}{dt}\right) d\left(\frac{di}{dt}\right) = -\frac{i\,di}{LC}.$$

By integrating we have

$$\left(\frac{di}{dt}\right)^2 = -\frac{i^2}{LC} + c.$$

$$\frac{di}{dt} = \pm\sqrt{c - \frac{i^2}{LC}}.$$

The variables may be readily separated, thus:

$$\frac{di}{\sqrt{c - \frac{i^2}{LC}}} = dt. \tag{228}$$

The integral of (228) is obtained by the formula of integration,

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a}.$$

Upon integration it becomes

$$\sin^{-1}\frac{i}{\sqrt{cLC}} = \frac{t}{\sqrt{LC}} + c_1.$$

Taking the sine of each member and writing c' for $\sqrt{cLC}$,

$$i = c' \sin\left(\frac{t}{\sqrt{LC}} + c_1\right). \tag{229}$$

The only two variables in this equation are i and t, and the current is seen to be a sine-function of the time. When the current is a maximum, the sine is unity and we have

$$I = c'.$$

If the time is reckoned from the point where the current is zero, $t = 0$ when $i = 0$, and we have

$$c_1 = 0.$$

Substituting these values for the constants c' and c_1, we have

$$i = I \sin\frac{1}{\sqrt{LC}}\, t. \tag{230}$$

In an harmonic function, as this, the coefficient of the variable t is the angular velocity which we designate by ω. Equation (230) then becomes

$$i = I \sin \omega t. \tag{231}$$

We have, then, the necessary conditions by which the self-induction and capacity will just neutralize each other at every point of time. The current must be a simple sine-function of the time, and the self-induction and capacity must have such values that $\omega = \dfrac{1}{\sqrt{LC}}$. By no other conditions, with self-induction and capacity in a circuit, can

the instantaneous values of the current be the same as though the capacity and self-induction were absent.

If the heating effect, or any effect which depends upon $\int i^2\,d\,t$, *in a circuit is the same when the self-induction and capacity are present as it is when they are absent, the impressed E. M. F. must be a simple harmonic E. M. F.*—Since we have found that there is no possible relation between L and C, so that the *instantaneous* values of the current are unchanged by their introduction into a circuit with an impressed E. M. F. which is not an harmonic function, it is interesting to inquire whether any relation can be given L and C so that the *energy* spent in the conductor in a given time is the same before as after the introduction of L and C.

Before attempting to investigate such a relation, it will be well to first consider some different classes of current curves, then ascertain the $\int i^2\,d\,t$ effect for some particular current curves, and afterwards consider the energy of any periodic curve whatever.

Fig. 37 represents a curve which has an equal area above and below the axis every period. This means that the in-

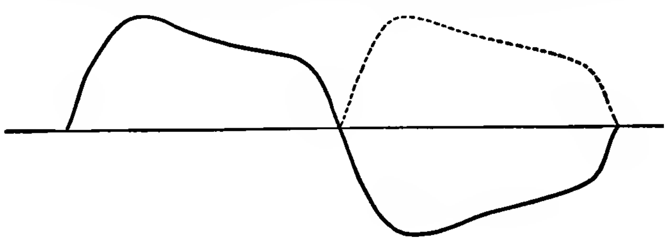

FIG. 37.

tegral $\int i\,d\,t$ for one period is zero, that is, the quantity of electricity which flows each period in the positive direction is equal to that which flows in the negative direction. Moreover, if the lower half of the current curve is inverted

and represented by the dotted line, it is an exact repetition of the first half of the curve. This curve may represent the type of current curves given by alternating generators in circuits with resistance, self-induction, and capacity; for, it is evident that, as the armature revolves, the number of lines introduced into the circuit every period equals those taken from the circuit. Now, the quantity of current which flows is strictly proportional to the change in the number of lines threading the circuit. This is equivalent to saying that the quantity which flows in the positive direction is exactly equal to the quantity flowing in the negative direction, or the total algebraic quantity per period is zero. Now, if the generator is exactly symmetrical, the current curve in the second half of the period is, if inverted, an exact repetition of the curve in the first half. Any irregularities in the symmetry of the machine might cause slight differences in the two parts of the curve, but hardly enough to prevent this curve from representing the type of curves given by alternating machines. During every complete revolution of the armature, the total algebraic quantity of current flowing must be rigorously equal to zero, no matter how many irregularities there may be in the machine; for, the number of lines introduced into the circuit exactly equals those subtracted from the circuit, because after a complete revolution the number of lines is the same as at the start. It is possible that adjacent positive and negative areas may be unequal in a multipolar machine, due to some irregularity in the machine, but after a complete revolution of the armature the sum of the positive areas equals the sum of the negative.

Fig. 38 represents a current curve which has equal areas above and below the axis every period, but the negative area, when inverted, is not necessarily a repetition of the positive area. This represents the type of current curve

when there is a non-leaky condenser in the circuit, since the total algebraic flow here is necessarily equal to zero.

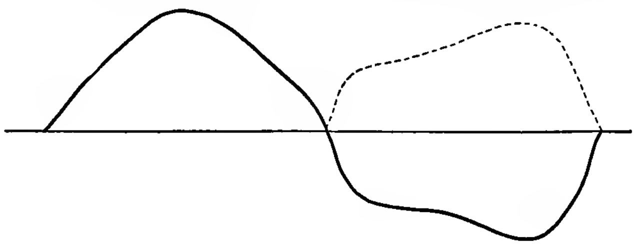

FIG. 38.

Fig. 39 represents a current curve in which the negative area is neither equal to the positive area nor symmetrical with it when inverted.

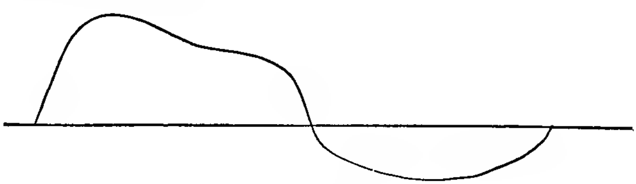

FIG. 39.

It is interesting to inquire whether the $\int i^2\,dt$ effect is the same in a circuit while the current flows in the positive direction as it is while flowing in the negative direction. We can see that it is the same for a current of the type represented in Fig. 37, for, squaring the ordinate at each point and drawing a new curve, *b*, Fig. 40, the $\int i^2\,dt$ effect is proportional to the areas of this new curve. Since the current curves *a*, *a* are exact repetitions, these areas, *b*, *b*, are identical, and the $\int i^2\,dt$ effect is the same when the current is positive as it is when negative.

Let us inquire how this is for a current of the type of

Fig. 38, where the areas are equal, that is, the $\int i\,dt$ is the same for positive as for negative current, but the negative

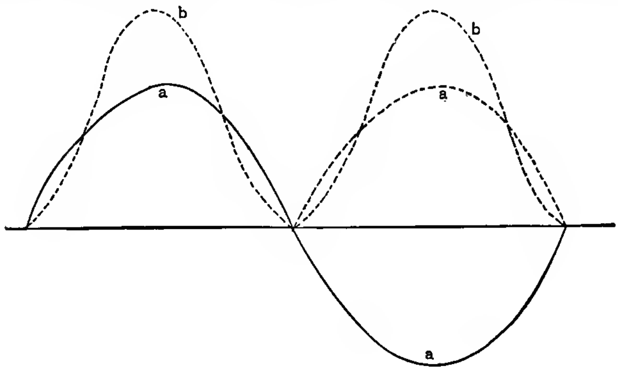

Fig. 40.

part, when inverted, is not an exact repetition of the positive part. In Fig. 41 the areas between the axis and the cur-

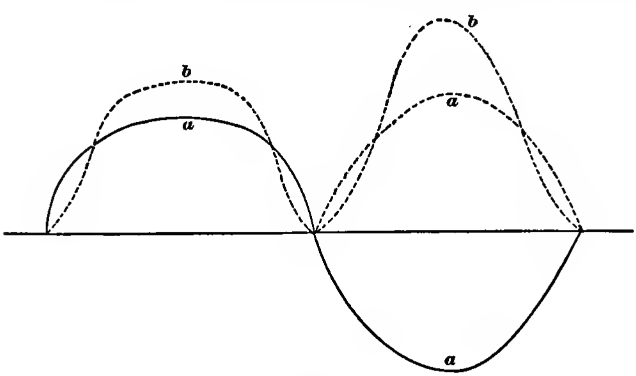

Fig. 41.

rent curve a, a are equal for each half period. The curve b, b is drawn by squaring each ordinate of the curve a. The areas b, b represent the $\int i^2 dt$ effect, and we wish to find whether they are equal.

Illustration from a Particular Case.—To show that this $\int i^2 dt$ effect is not necessarily the same when the current is positive as when it is negative, it will suffice to take one

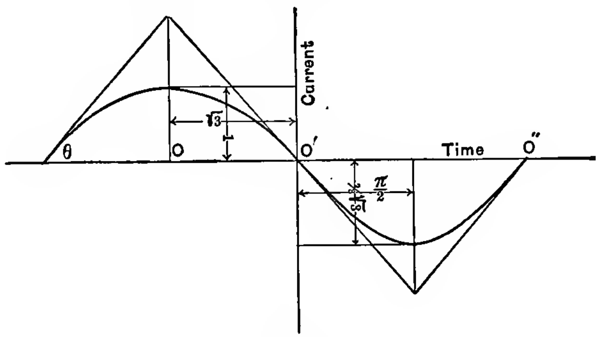

FIG. 42.

particular case of a current curve. Suppose the positive curve is a parabola (Fig. 42) whose equation, referred to O as an origin, is

(232) $$t^2 = -3i + 3.$$

Suppose that the negative curve is a sine-curve whose equation, referred to O'' as origin, is

(233) $$i = \tfrac{2}{3}\sqrt{3} \sin t.$$

It is easily shown that the areas of these curves are equal.

$$\text{Area parabola} = \tfrac{2}{3}\,[\text{base} \times \text{height}].$$

One-half of the base of the parabola is found by making $i = 0$ in equation (232) and finding the value of t.

Therefore, $\quad \tfrac{1}{2}\,\text{base} = \sqrt{3},$

$$\text{Base} = 2\sqrt{3}.$$

The height is found by making $t = 0$ and finding the value of i.

Therefore, Height = unity.

(234) Hence Area parabola $= \frac{2}{3}$ [base $\times$ height] $= \frac{4}{3}\sqrt{3}$.

The area of the sine-curve is equal to the mean ordinate multiplied by the base; therefore

$$\text{Area sine-curve} = \text{mean ordinate} \times \pi.$$

The mean ordinate of a sine-curve equals twice the maximum ordinate divided by π. [See p. 37.] By equation (233), the maximum ordinate equals $\frac{2}{3}\sqrt{3}$ and, therefore, mean ordinate $= \frac{4}{3\pi}\sqrt{3}$, and

$$\text{Area sine-curve} = \tfrac{4}{3}\sqrt{3}, \tag{235}$$

which is the same as the area of the parabola given in (234) above. Moreover, the tangents of the angles which these two curves make at the point O with the axis are equal, and the curves consequently blend into one another without any abrupt change in continuity. This is easily shown as follows: Differentiating (232) and (233) respectively, we have

$$\frac{d\,i}{d\,t} = -\tfrac{2}{3}t = \tan\theta. \tag{236}$$

$$\frac{d\,i}{d\,t} = \tfrac{2}{3}\sqrt{3}\cos t = \tan\theta'. \tag{237}$$

Making $t = \sqrt{3}$ in (236), we have the tangent of the inclination of the parabola at the point O. Making $t = -\pi$ in (237), we have the tangent of the inclination of the sine-curve at the point O. These values, it is noticed, reduce (236) and (237), respectively, to $\tan\theta = \tan\theta' = -\frac{2}{3}\sqrt{3}$, which is the value of the tangent of inclination of either curve at the point O.

It remains to find the $\int i^2\,dt$ for each of these curves. By transposition, the equation of the parabola (232) is

$$i = 1 - \frac{t^2}{3}.$$

By squaring,

$$i^2 = 1 - \frac{2}{3}t^2 + \frac{t^4}{9}.$$

$$\int i^2\,dt = \int dt - \frac{2}{3}\int t^2\,dt + \frac{1}{9}\int t^4\,dt.$$

Integrating between the limits $-\sqrt{3}$ and $\sqrt{3}$, we have

$$\int_{-\sqrt{3}}^{\sqrt{3}} i^2\,dt = \Big[t - \tfrac{2}{9}t^3 + \tfrac{1}{45}t^5\Big]_{-\sqrt{3}}^{\sqrt{3}}.$$

$$\int_{-\sqrt{3}}^{\sqrt{3}} i^2\,dt = 2\sqrt{3} - \frac{4\sqrt{3}^3}{9} + \frac{2\sqrt{3}^5}{45} = \frac{16}{15}\sqrt{3}.$$

This is the $\int i^2\,dt$ effect for the parabola.

For the sine-curve the equation is

$$i = \tfrac{2}{3}\sqrt{3}\sin t.$$

$$\int i^2\,dt = \frac{4}{3}\int \sin^2 t.$$

Integrating between the limits 0 and π,

$$\int_0^{\pi} i^2\,dt = \frac{4}{3} \times \Big[\frac{t}{2} - \frac{1}{2}\sin t\cos t\Big]_0^{\pi} = \frac{4}{3} \times \frac{\pi}{2} = \frac{2}{3}\pi.$$

This gives the $\int i^2\,dt$ effect for the sine-curve. Hence we find that, although the area of the current curve is the same for the positive and the negative current—that is, the total algebraic quantity of flow is zero—yet the $\int i^2\,dt$ effect is

different in the positive and negative directions. In the case supposed, the ratio of the two effects is

$$\frac{\frac{2}{3}\pi}{\frac{16}{15}\sqrt{3}} = 1.135.$$

This may afford an explanation for the fact that in many cases one carbon of an alternating-current arc lamp is consumed more rapidly than the other, depending upon the way it is connected up.

General Proof.—Let us now return to the consideration of the energy in a conductor when any periodic E. M. F. is applied, and ascertain whether there is any condition by which self-induction and capacity may be introduced into the circuit without changing the energy or $\int i^2\,dt$ effect.

The energy expended in a conductor is proportional to $\int i^2\,dt$. When the E. M. F. in the circuit is

$$e = \sum_{E,b,\theta} E \sin(b\,\omega\,t + \theta), \quad \text{[see (221),]}$$

which represents any periodic E. M. F., it has been shown that the current is

$$(238) \qquad i = \sum_{E,b,\theta} \frac{E}{\sqrt{R^2 + \left(\frac{1}{C b\,\omega} - L b\,\omega\right)^2}} \sin\left\{b\,\omega t + \theta + \tan^{-1}\left(\frac{1}{C R b\,\omega} - \frac{L b\,\omega}{R}\right)\right\},$$

neglecting the complementary function [see (223)]. And, when there is neither self-induction nor capacity, the current is

$$(239) \qquad i_0 = \sum_{E,b,\theta} \frac{E}{R} \sin(b\,\omega\,t + \theta).$$

If we put

(240) $$I = \frac{E}{\sqrt{R^2 + \left(\frac{1}{Cb\omega} - Lb\omega\right)^2}},$$

(241) $$I_0 = \frac{E}{R},$$

and $$\alpha = \theta + \tan^{-1}\left(\frac{1}{CRb\omega} - \frac{Lb\omega}{R}\right),$$

we may abbreviate (238) and (239) as follows:

(242) $$i = \sum I \sin(b\omega t + \alpha).$$

(243) $$i_0 = \sum I_0 \sin(b\omega t + \theta).$$

The subscript $_0$ indicates the absence of self-induction and capacity. Remembering that the energy is proportional to $\int i^2\,dt$, we have

(244) $$W = \int i^2\,dt = \int \left[\sum I \sin(b\omega t + \alpha)\right]^2 dt,$$

and

(245) $$W_0 = \int i_0^2\,dt = \int \left[\sum I_0 \sin(b\omega t + \theta)\right]^2 dt,$$

where W is proportional to the energy expended in the circuit with L and C, and W_0 bears the same relation to the energy when they are absent. In order to find what relation must exist between L and C to cause the energy expended during a certain time to be the same in both cases, we must integrate (244) and (245) between the same limits of time, and equate them. In order to simplify (244) and (245), express as follows:

(246) $$W = \int \left[I' \sin(b_1\omega t + \alpha_1) + I'' \sin(b_2\omega t + \alpha_2) + I''' \text{ etc.}\right]^2 dt.$$

(247) $$W_0 = \int \left[I_0' \sin(b_1\omega t + \theta_1) + I_0'' \sin(b_2\omega t + \theta_2) + I_0''' \text{etc.}\right]^2 dt.$$

Since the square of any polynomial is equal to the sum of the squares of each term separately plus twice the product of each term by every other term, we have as a result to find the integrals of only two forms, thus:

(248) $$\int \sin^2 (b\,\omega\,t + \alpha)\,d\,t, \quad \text{and}$$

(249) $$\int \sin (b_1\,\omega\,t + \alpha_1) \sin (b_2\omega\,t + \alpha_2).$$

If the limits are taken from $t = 0$ to $t = T$, a complete period,—the E. M. F. being periodic with a period $T = \frac{2\,\pi}{\omega}$, —it can be shown that all the integrals of the form of (249) vanish; for, expressing the sine of the sum of two angles in terms of the sines of the angles themselves,

(250) $$\sin(b_1\,\omega\,t + \alpha_1) = \sin b_1\omega t \cos \alpha_1 + \sin \alpha_1 \cos b\,\omega\,t,$$

and

(251) $$\sin(b_2\omega\,t + \alpha_2) = \sin b_2\,\omega\,t \cos \alpha_2 + \sin \alpha_2 \cos b_2\,\omega\,t.$$

Multiplying (250) and (251), we obtain terms of the following forms:

(252) $$\int \sin b_1\,\omega t \cos b_2\,\omega t\,d\,t,$$

(253) $$\int \cos b_1\,\omega t \cos b_2\,\omega\,t\,d\,t$$

(254) $$\int \sin b_1\,\omega\,t \sin b_2\,\omega\,t\,d\,t,$$

which are to be integrated between the limits 0 and T, or $\frac{2\,\pi}{\omega}$. Substituting for $b_1\omega\,t$, $a\,x$, and for $b_2\omega\,t$, $b\,x$, we have made the integral in (249) depend upon the three forms,

(255) $$\int_0^{2\,\pi} \sin a\,x \cos b\,x\,d\,x,$$

(256) $$\int_0^{2\pi} \cos ax \cos bx\,dx,$$

(257) $$\int_0^{2\pi} \sin ax \sin bx\,dx.$$

To show that each of these three forms vanishes between the limits zero and 2π, we can reduce as follows:

(258) $$\int_0^{2\pi} \sin ax \cos bx dx = \frac{1}{2}\int_0^{2\pi} \sin(a+b)x\,dx + \frac{1}{2}\int_0^{2\pi} \sin(a-b)x dx = -\frac{1}{2}\Big[\frac{\cos(a+b)x}{a+b} + \frac{\cos(a-b)x}{a-b}\Big]_0^{2\pi} = 0.$$

(259) $$\int_0^{2\pi} \cos ax \cos bx dx = \frac{1}{2}\int_0^{2\pi} \cos(a+b)x\,dx + \frac{1}{2}\int_0^{2\pi} \cos(a-b)x dx = \frac{1}{2}\Big[\frac{\sin(a-b)x}{a-b} + \frac{\sin(a+b)}{a+b}\Big]_0^{2\pi} = 0.$$

(260) $$\int_0^{2\pi} \sin ax \sin bx dx = \frac{1}{2}\int_0^{2\pi} \cos(a-b)x\,dx - \frac{1}{2}\int_0^{2\pi} \cos(a+b)\,x\,dx = \frac{1}{2}\Big[\frac{\sin(a-b)x}{a-b} - \frac{\sin(a+b)x}{a+b}\Big]_0^{2\pi} = 0.$$

Since, therefore, the integral in (249) is zero in every case, we have only to find the integral expressed in (248). This is

(261) $$\int_0^{\frac{2\pi}{\omega}} \sin^2(b\,\omega t + \alpha)dt = \Big[\frac{b\,\omega t + \alpha}{2b\,\omega} - \frac{1}{4b\,\omega}\sin 2(b\,\omega t + \alpha)\Big]_0^{\frac{2\pi}{\omega}} = \frac{\pi}{\omega} = \frac{T}{2},$$

which is obtained by the formula

$$\int \sin^2 x\,dx = \frac{x}{2} - \frac{1}{4}\sin 2x,$$

upon replacing x by $b\omega t + \alpha$, and dx by $b\omega\, dt$. Returning to equations (246) and (247), and replacing the value of the integral in (261), as determined, we have now found the values of W and W_0 in equations (246) and (247) to be

$$W = \left[I'^2 + I''^2 + I'''^2 + \text{etc.}\right]\frac{T}{2},$$

and $$W_0 = \left[I_0'^2 + I_0''^2 + I_0'''^2 + \text{etc.}\right]\frac{T}{2},$$

or $$W = \frac{T}{2}\sum I^2,$$

$$W_0 = \frac{T}{2}\sum I_0^2.$$

Equating W and W_0, as before explained, to determine the condition necessary to make the energy the same, we obtain

(262) $$\sum I^2 = \sum I_0^2,$$

which, written in full, is

(263) $$\sum \frac{E^2}{R^2 + \left(\frac{1}{Cb\omega} - Lb\omega\right)^2} = \sum \frac{E^2}{R^2}.$$

[See (240) and (241).] This equation expresses the relation which must be true if the $\int i^2\, dt$ effect is the same when the self-induction and capacity are present as it is when they are absent. This equation expressed without the sign of summation is

(264) $$\frac{E_1^2}{R^2 + \left(\frac{1}{Cb_1\omega} - Lb_1\omega\right)^2} + \frac{E_2^2}{R^2 + \left(\frac{1}{Cb_2\omega} - Lb_2\omega\right)^2}$$

$$+ \text{etc.} = \frac{E_1^2}{R^2} + \frac{E_2^2}{R^2} + \text{etc.}$$

It is evident that the parenthesis in the denominator of each term of the first member, being squared, is always positive no matter what values L and C may have. Each term, then, of the first member is less than the corresponding term in the second member, unless the expression in the parenthesis is zero. And in order that the first member shall be as large as the second member, each parenthesis must be separately equal to zero; that is, we must have

$$b_1\omega = \frac{1}{\sqrt{LC}},$$

$$b_2\omega = \frac{1}{\sqrt{LC}},$$

and $$b_3\omega = \frac{1}{\sqrt{LC}}, \quad \text{etc.}$$

Therefore $b_1 = b_2 = b_3 =$ etc. But this condition is equivalent to saying that the impressed E. M. F. can only be a simple harmonic E. M. F., and that we must have the relation $\omega = \frac{1}{\sqrt{LC}}$ in order to have the $\int i^2\,dt$ effect the same in a circuit when the self-induction and capacity are present as when they are absent. There is, then, no relation between the self-induction and capacity which can be given that will make the $\int i^2\,dt$ effect the same in a circuit when they are present as when they are absent, if the impressed E. M. F. is not an harmonic E. M. F.

CHAPTER XII.

CIRCUITS CONTAINING DISTRIBUTED CAPACITY AND SELF INDUCTION. GENERAL SOLUTION.*

CONTENTS:—Derivation of the differential equations for circuits containing distributed capacity only. This equation extended so as to represent a particular case of distributed capacity and self-induction. Differential equation for E. M. F. is of the same form as that for current. The general solutions of the differential equations. Particular assumption of harmonic E. M. F. Constants of the general equation determined under this assumption; first, from the exponential solution; second, from the sine solution. Current determined from the E. M. F. equation.

IN former chapters the only capacity considered has been that due to a condenser placed at some particular point of the circuit, thus introducing an actual break in the continuity of the conducting metal. It is possible to have the effects of capacity without thus introducing a condenser into the circuit. The problem of the propagation of the electric current in a cable containing distributed static capacity was first discussed by Sir William Thomson, and

* The purpose in writing this book has been to give concisely such principles as are necessary for a clear understanding of alternate-current phenomena, and to make the work one connected unit, dealing with the various problems in turn, so that no portion could be omitted without interfering with the logical sequence. This and the following chapter constitute, however, a separate discussion which may be read alone, and without which the rest of the book is logically complete.

NOTE.—The authors' thanks are due to Prof. Merritt for calling attention to certain discrepancies in the signs of some of the equations from 273 to 317 in the first edition. These discrepancies did not affect the results and have been rectified in the present edition.

afterwards by Mascart and Joubert,* Blakesley,† and others. The solution for the variation in the current and potential at different points of a conductor containing self-induction as well as distributed capacity was given by the authors in the *American Journal of Science*‡ and some of the effects of the self-induction noted, and a fuller discussion was given in the *London Electrician.*§

When a current of electricity flows in a wire, the potential of the wire at any point is generally different from the potential of the surrounding medium, and in order that this potential may be different it is necessary that the exterior surface of the wire should become charged with a certain amount of electricity. A portion of the current, then, as it flows along the wire, is used to charge the surface of the wire. Indeed, the wire must be charged with its proper amount before the current can flow on to more distant parts of the circuit. It is evident, then, that the larger the capacity of the wire to hold a charge, the greater will be its effect in modifying the flow of current. The capacity per unit length of the wire (the wire being regarded as one plate of the condenser) depends upon its superficial area and upon the thickness of the dielectric (usually between it and the conducting earth near it), as well as upon its nature. In Fig. 43 is represented the longitudinal section of a cable, A being the conducting wire and BB the insulating sheath around it. Suppose it to be submersed in water; the other conductor is the water, which, with the wire, forms the condenser.

Let the capacity of a unit length of the wire be denoted by C, and the capacity of an element PQ, whose length is

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* Mascart and Joubert, Leçons sur l'électricité et le magnétisme, Vol. I., § 233.

† T. H. Blakesley, Alternating Currents of Electricity, Chap. VIII.

‡ Vol. XLIV., page 389.

§ Vol. XXIX., pages 619 and 634.

dx, by Cdx. Let R denote the resistance of a unit length of the wire. The resistance of the element PQ is Rdx. Suppose a current i is flowing across the section of the wire at P in the positive direction indicated by the arrow. Let the potential of P at that instant be denoted by e.

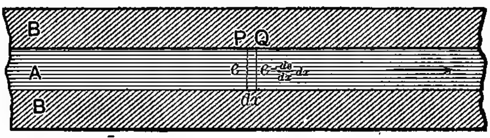


FIG. 43.—LONGITUDINAL SECTION OF CABLE.

Since the current always flows from the higher to the lower potential, the potential at Q, the other end of the element, must be less than that at P, and the potential therefore diminishes in the positive direction. This fall of potential from P to Q is denoted by $-\frac{de}{dx}dx$. By Ohm's law the current i, at any moment through the element PQ, equals the difference of potential divided by the resistance, and is, therefore,

$$(265) \qquad i = -\frac{\frac{de}{dx}dx}{Rdx} = -\frac{1}{R}\frac{de}{dx}.$$

If the current remained constant, having this value i all the time, the potential of the element and its charge would continually remain the same, and the flow of electricity across the section Q would be the same as that at P, since as much must flow out from as into the element, unless the charge of the element be changed. Now, considering that the current does not remain constant but changes every moment of time, the potential e of the element, and consequently its charge, must change with the time. When the charge changes, it means that more electricity is flowing

into than out from the element, or *vice versa*, and consequently the flow of current across P is different from that across Q by just such an amount as the element gains or loses. The current at Q is then denoted by $i + \frac{d\,i}{d\,x}\,d\,x$.

Let the quantity flowing across the section P, in the time $d\,t$, be denoted by $d\,Q$, and that across the section Q by $d\,Q - d\,q$, where $d\,q$ is the change in the charge of the element in the time $d\,t$. The quantity of electricity flowing across the section P is equal to the current flowing at P multiplied by the time; that is,

$$d\,Q = i\,d\,t, \quad \text{or} \quad \frac{d\,Q}{d\,t} = i. \tag{266}$$

Similarly the flow across Q is the current flowing at Q multiplied by the time; that is,

$$d\,Q - dq = \left(i + \frac{d\,i}{d\,x}d\,x\right)d\,t. \tag{267}$$

Subtracting (267) from (266), we obtain

$$\frac{d\,q}{d\,t} = -\frac{d\,i}{d\,x}d\,x. \tag{268}$$

This equation may be interpreted to mean that the rate of change of the charge on the element is equal to the difference of the currents flowing into the element and out from it. We might at once have written this equation from this consideration.

The charge of the element, as of any condenser, is equal to its capacity multiplied by its potential. The charge being denoted by q, the potential by e, and the capacity, as stated above, by $C\,d\,x$, we have

$$q = C\,e\,d\,x. \tag{269}$$

The rate of change of the charge with the time is, by differentiation,

$$\frac{dq}{dt} = C\frac{de}{dt}dx. \tag{270}$$

Equating this result to equation (268), we have

$$-\frac{di}{dx} = C\frac{de}{dt}. \tag{271}$$

Equations (265) and (271) are the differential equations which are sufficient to determine the problem of the propagation of the current along a cable containing distributed capacity such as that described, when the impressed E. M. F. of the source is known. The solution of these equations may be obtained for the most general case, although the arbitrary constants of integration can only be determined in certain particular cases where the impressed E. M. F. is known.

When the impressed E. M. F. is harmonic and equal to $e = E \sin \omega t$, the arbitrary constant may be found.

These two differential equations may be expressed as a single equation by differentiating (265) with respect to x and equating to (271), thus:

$$\frac{di}{dx} = -\frac{1}{R}\frac{d^2e}{dx^2},$$

(272) and, therefore, $\dfrac{d^2e}{dx^2} = CR\dfrac{de}{dt}.$

In the foregoing discussion no account has been taken of the self-induction of the circuit, but it necessarily has a certain effect upon the flow of the current which it would be well, if possible, to consider. The effect of the self-induction must be felt as a back E. M. F. opposing the current and depending upon its rate of change. We shall assume that the back E. M. F. per unit length of the con-

ductor is equal to the rate of change of the current, multiplied by a constant; that is, it is equal to $L \frac{d i}{d t}$. In some cases this assumption may approximately represent the true effect of self-induction, and it is thought that this particular assumption may show the nature of the effect of self-induction even in cases where the assumption is not justifiable.

Instead of leaving equation (265) as it stands, therefore, without taking into account the effect of the back E. M. F. of self-induction, we may introduce this effect into the equation by subtracting from the difference of potential between P and Q, viz., $-\frac{d e}{d x} d x$, the internal E. M. F. of self-induction, $L \frac{d i}{d t} d x$, and so may write, still in accordance with Ohm's law,

$$(273) \quad i = \frac{-\frac{d e}{d x} d x - L \frac{d i}{d t} d x}{R d x} = -\frac{1}{R} \frac{d e}{d x} - \frac{L}{R} \frac{d i}{d t}.$$

The relation in equation (271) is not changed by the consideration of the self-induction, and these two equations, (271) and (273), are sufficient to determine the problem of the flow of current, taking into account both the capacity and self-induction. These equations, now containing four variables, may be expressed as two differential equations containing three variables by eliminating first i and then e.

After transposing and arranging, we may write (271) and (273)

$$(274) \qquad C \frac{d e}{d t} + \frac{d i}{d x} = 0.$$

$$(275) \qquad \frac{d e}{d x} + L \frac{d i}{d t} + R i = 0.$$

Operating upon (274) by $\frac{d}{d\,t}$, that is, differentiating with respect to t, we obtain

$$C\frac{d^2 e}{d\,t^2}+\frac{d\left(\frac{d\,i}{d\,x}\right)}{d\,t}=0. \tag{276}$$

Operating upon (275) by $\frac{1}{L}\frac{d}{d\,x}$, we find

$$\frac{1}{L}\frac{d^2 e}{d\,x^2}+\frac{d\left(\frac{d\,i}{d\,t}\right)}{dx}+\frac{R}{L}\frac{d\,i}{dx}=0. \tag{277}$$

From (276) subtracting (277), we have

$$C\frac{d^2 e}{d\,t^2}-\frac{1}{L}\frac{d^2 e}{d\,x^2}-\frac{R}{L}\frac{d\,i}{d\,x}=0. \tag{278}$$

Substituting here the value of $\frac{d\,i}{d\,x}$, namely, $-\,C\frac{d\,e}{d\,t}$, in (274), we eliminate i and finally have for the differential equation of potential

$$\frac{d^2 e}{d\,x^2}-LC\frac{d^2 e}{d\,t^2}-RC\frac{d\,e}{d\,t}=0. \tag{279}$$

To eliminate e from (274) and (275), operate upon (274) by $\frac{d}{dx}$, and upon (275) by $C\frac{d}{d\,t}$, and we have

$$C\frac{d\left(\frac{d\,e}{d\,t}\right)}{dx}+\frac{d^2 i}{dx^2}=0, \tag{280}$$

$$\text{and}\quad C\frac{d\left(\frac{d\,e}{d\,x}\right)}{d\,t}+L\,C\frac{d^2 i}{d\,t^2}+RC\frac{d\,i}{d\,t}=0. \tag{281}$$

Subtracting (281) from (280), we have the differential equation for current

$$\frac{d^2 i}{d x^2} - LC \frac{d^2 i}{d t^2} - RC \frac{d i}{d t} = 0. \tag{282}$$

It is evident from the similarity of equations (279) and (282) that the integral current equation will be the same as the integral potential equation, except for the arbitrary constants that enter in integration.

To Find the Solutions of the Differential Equations.

Assume that the solutions of the pair of differential equations (274) and (275) are

$$e = k \epsilon^{mx + nt}, \tag{283}$$

$$\text{and} \quad i = \epsilon^{mx + nt}, \tag{284}$$

where m, n, and k are constants which must be determined, and x is the distance from the source of E. M. F. These constants may be determined by differentiation so that the equations satisfy the differential equations (274) and (275), and are, therefore, correct solutions. Differentiating (283) and (284) with regard to x and t, we obtain

$$\frac{d e}{d x} = m k \epsilon^{mx + nt};$$

$$C \frac{d e}{d t} = n k C \epsilon^{mx + nt};$$

$$\frac{d i}{d x} = m \epsilon^{mx + nt};$$

$$L \frac{d i}{d t} = L n \epsilon^{mx + nt}.$$

Substituting these values in (274) and (275), we obtain the simultaneous equations

$$(285) \qquad n\,k\,C + m = 0,$$

$$(286) \qquad \text{and} \quad m\,k + L\,n + R = 0.$$

If these equations are satisfied, the differential equations are likewise satisfied. Solving for m and n, we find

$$(287) \qquad m = -\frac{C\,k\,R}{C\,k^2 - L}.$$

$$(288) \qquad n = +\frac{R}{C\,k^2 - L}.$$

Substituting these constants in (283) and (284), we have

$$(289) \qquad e = k\,\epsilon^{\frac{R}{Ck^2 - L}(t - C\,k\,x)},$$

$$(290) \qquad \text{and} \quad i = \epsilon^{\frac{R}{Ck^2 - L}(t - C\,k\,x)}.$$

These equations are solutions of equations (274) and (275), and they may be easily verified by differentiation. But a more general solution might be obtained by assuming the E. M. F., e, to be a sum of several terms such as that already assumed, thus:

$$(291) \; e = h_1\,k_1\,\epsilon^{m_1x + n_1t} + h_2\,k_2\,\epsilon^{m_2x + n_2t} + \ldots = \sum_{h,\,k.} h\,k\,\epsilon^{mx + nt}.$$

$$(292) \; i = h_1\,\epsilon^{m_1x + n_1t} + h_2\,\epsilon^{m_2x + n_2t} + \ldots = \sum_{h.} h\,\epsilon^{mx + nt}.$$

Determining m and n as before, (291) and (292) may be expressed

$$(293) \qquad e = \sum_{h,\,k.} h k \epsilon^{\frac{R}{Ck^2 - L}(t - Ckx)}.$$

$$(294) \qquad i = \sum_{h,\,k.} h \epsilon^{\frac{R}{Ck^2 - L}(t - Ckx)}.$$

These equations may also be verified by differentiation and found to satisfy the differential equations (274) and (275), and they are the complete integrals of those differential equations. If we know how the current or the potential varies with the time at any one point of the wire, the arbitrary constants h and k can be determined, and we have the complete solution of the problem, and are enabled to tell the potential or current at every point of the wire at any time.

HARMONIC E. M. F.

The general solutions, (293) and (294), hold true in case the constants to be determined are real or imaginary; if they are imaginary, the equations may be transformed into a real form consisting of some function of the sine.

Suppose the cable before described is indefinitely long, and that at the point P (arbitrarily selected as the zero point of the wire, the positive direction being indicated by the arrow) the potential is caused to vary harmonically with the time and is always equal to

$$(295) \qquad e = E \sin \omega t.$$

Since equation (293) expresses the E. M. F. at every point of the wire and at every moment of time, we may, by making $x = 0$ in that equation, find an expression giving

the potential at the origin at every moment of time. This expression is

$$(296) \qquad e = \sum_{h,\,k.} h\,k\,\epsilon^{\frac{R}{Ck^2-L}t}$$

But, since we have supposed this potential to be harmonic, we may equate equation (295) to (296) and determine the constants h and k so as to make the expressions identical. Equating the equations thus, we have

$$(297) \qquad E \sin \omega t = \sum_{h,\,k.} hk\,\epsilon^{\frac{R}{Ck^2-L}t}.$$

In order to determine the constants, we write the sine in its exponential form, thus:

$$\sin \omega t = \frac{\epsilon^{+j\omega t} - \epsilon^{-j\omega t}}{2j},$$

where j stands for $\sqrt{-1}$. (See equation (109), Chapter VII., with footnote.) We may, therefore, substitute in (297) the exponential value of the sine and write only two terms of the summation, thus:

$$(298) \qquad e = E \sin \omega t =$$

$$\frac{E}{2j}\epsilon^{j\omega t} - \frac{E}{2j}\epsilon^{-j\omega t} = h_1 k_1\,\epsilon^{\frac{R}{Ck_1^2-L}t} + h_2 k_2\,\epsilon^{\frac{R}{Ck_2^2-L}t}.$$

This becomes an identity if

$$(299) \qquad h_1 k_1 = \frac{E}{2j}, \quad \text{and} \quad h_2 k_2 = -\frac{E}{2j}.$$

$$(300) \quad \text{Also, if } j\omega = \frac{R}{Ck_1^2 - L}, \quad \text{and} \quad -j\omega = \frac{R}{Ck_2^2 - L}.$$

Solving equations (300) for k_1 and k_2, we have

(301) $$k_1 = \pm \frac{\sqrt{-j}}{\sqrt{C\omega}} \sqrt{L\omega j + R}.$$

(302) $$k_2 = \pm \frac{\sqrt{-j}}{\sqrt{C\omega}} \sqrt{L\omega j - R}.$$

Since all the constants are found to be imaginary, this imaginary exponential expression for the E. M. F. should be transformed into a real expression involving some function of the sine. This sine-function may be found by continuing the method already indicated. The next step necessary is to transform the complex imaginary values of k by a rather laborious process until the imaginary j is removed from under the radical sign.

It will be evident that the following equations are identically true, either by squaring each member and seeing that they are identical, or by supposing either R or L to be zero, when they reduce to an identity.

(303) $$\sqrt{L\omega j + R} = \sqrt{\frac{(R^2 + L^2\omega^2)^{\frac{1}{2}} + R}{2}} + j\sqrt{\frac{(R^2 + L^2\omega^2)^{\frac{1}{2}} - R}{2}}.$$

(304) $$\sqrt{L\omega j - R} = \sqrt{\frac{(R^2 + L^2\omega^2)^{\frac{1}{2}} - R}{2}} + j\sqrt{\frac{(R^2 + L^2\omega^2)^{\frac{1}{2}} + R}{2}}.$$

Substituting these expressions in (301) and (302), and writing Im for the impedance $(R^2 + L^2\omega^2)^{\frac{1}{2}}$, we have

$$(305)\quad k_1 = \pm\left\{\frac{\sqrt{j}}{\sqrt{2C\omega}}\sqrt{Im - R} + \frac{\sqrt{-j}}{\sqrt{2C\omega}}\sqrt{Im + R}\right\}.$$

$$(306)\quad k_2 = \pm\left\{\frac{\sqrt{j}}{\sqrt{2C\omega}}\sqrt{Im + R} + \frac{\sqrt{-j}}{\sqrt{2C\omega}}\sqrt{Im - R}\right\}.$$

Since we know that

$$\sqrt{+j} = \frac{1+j}{\sqrt{2}}, \quad \text{and} \quad \sqrt{-j} = \frac{1-j}{\sqrt{2}},$$

we may substitute these values in (305) and (306) and write

$$(307)\quad k_1 = \pm\left\{\frac{1}{2\sqrt{C\omega}}\left[\sqrt{Im - R} + \sqrt{Im + R}\right] + \frac{j}{2\sqrt{C\omega}}\left[\sqrt{Im - R} - \sqrt{Im + R}\right]\right\}.$$

$$(308)\quad k_2 = \pm\left\{\frac{1}{2\sqrt{C\omega}}\left[\sqrt{Im + R} + \sqrt{Im - R}\right] + \frac{j}{2\sqrt{C\omega}}\left[\sqrt{Im + R} - \sqrt{Im - R}\right]\right\}.$$

These values of k_1 and k_2 may be simplified, for we have the identities

$$(309)\quad \sqrt{Im - R} + \sqrt{Im + R} = \sqrt{2}\sqrt{Im + L\omega},$$

and

$$(310)\quad \sqrt{Im - R} - \sqrt{Im + R} = \sqrt{2}\sqrt{Im - L\omega}.$$

These may be verified by squaring both members. Upon the substitution of these values, the expressions for k_1 and k_2 become

$$(311) \quad k_1 = \pm \frac{1}{\sqrt{2}\, C\omega} \left\{ \sqrt{Im + L\omega} + j\sqrt{Im - L\omega} \right\}.$$

$$(312) \quad k_2 = \pm \frac{1}{\sqrt{2}\, C\omega} \left\{ \sqrt{Im + L\omega} - j\sqrt{Im - L\omega} \right\}.$$

Returning to equation (293) of E. M. F.'s and writing two terms of the summation, we have

$$(313) \quad e = h_1 k_1 \epsilon^{\frac{Rt}{Ck_1^2 - L} - \frac{Ck_1 Rx}{Ck_1^2 - L}} + h_2 k_2 \epsilon^{\frac{Rt}{Ck_2^2 - L} - \frac{Ck_2 Rx}{Ck_2^2 - L}}.$$

Substituting in (313) the values of h_1, k_1, h_2, k_2, $\frac{R}{Ck_1^2 - L}$, and $\frac{R}{Ck_2^2 - L}$ given in (299) and (300), we have

$$(314) \quad e = \frac{E}{2j} \left\{ \epsilon^{j\omega t - Ck_1 j\omega x} - \epsilon^{-j\omega t + Ck_2 j\omega x} \right\}.$$

Substituting in (314) the values of the constants k_1 and k_2, already given in equations (311) and (312), and factoring out the common factor $\epsilon^{\pm\sqrt{\frac{C\omega}{2}}(Im - L\omega)^{\frac{1}{2}} x}$, we have

$$(315) \quad e = E\epsilon^{\pm\sqrt{\frac{C\omega}{2}}(Im - L\omega)^{\frac{1}{2}} x}$$

$$\left\{ \frac{\epsilon^{j\omega t \mp j\sqrt{\frac{C\omega}{2}}(Im + L\omega)^{\frac{1}{2}} x} - \epsilon^{-j\omega t \pm j\sqrt{\frac{C\omega}{2}}(Im + L\omega)^{\frac{1}{2}} x}}{2j} \right\}.$$

Remembering the exponential value of the sine, equation (109), we may express (315) as a sine-function, thus:

$$(316) \quad e = E\,\epsilon^{\pm\left[\sqrt{\frac{C\omega}{2}}\sqrt{Im - L\omega}\right]x} \sin\left\{\omega t \pm \left[\sqrt{\frac{C\omega}{2}}\sqrt{Im + L\omega}\right]x\right\}.$$

This equation gives the value of the potential at any point of the conductor at any time. Its interpretation and discussion will be taken up in the following chapter.

Second Method of Obtaining the Solution.

We might have assumed the solution to be some function of the sine, since the potential at the origin is supposed to vary harmonically; and it is much easier to determine the constants if we do make such an assumption, inasmuch as we need not deal with imaginaries. Let us assume that the solution is of the form

$$e = h\,\epsilon^{rt+px} \sin(\beta t + \alpha x),$$

and determine the arbitrary constants α, β, h, r, and p so as to satisfy the differential equation (279). We see that the constant r must be zero, for when x is zero the E. M. F. is $E \sin \omega t$. Therefore $h = E$, and $\beta = \omega$. The constants p and α remain to be determined, and the E. M. F. is

$$(317) \qquad e = E\,\epsilon^{px} \sin(\omega t + \alpha x).$$

By differentiation we obtain (with E omitted)

$$\begin{array}{rl|l|l}
\dfrac{d^2 e}{dx^2} & = p^2 - \alpha^2 & \epsilon^{px}\sin(\omega t + \alpha x) + 2p\alpha & \epsilon^{px}\cos(\omega t + \alpha x) \\[2ex]
-CL\dfrac{d^2 e}{dt^2} & = +CL\omega^2 & \epsilon^{px}\sin(\omega t + \alpha x) & \\[2ex]
-CR\dfrac{de}{dt} & = & \qquad -CR\omega & \epsilon^{px}\cos(\omega t + \alpha x).
\end{array}$$

Equating the coefficients of the sine and cosine, separately, to zero, we obtain the simultaneous equations

$$p^2 - \alpha^2 + C L \omega^2 = 0,$$

and $$2 p \alpha - C R \omega = 0.$$

Solving these equations for p and α, we find that

(318) $$p = \pm \sqrt{\frac{C \omega}{2}} \sqrt{(R^2 + L^2 \omega^2)^{\frac{1}{2}} - L \omega},$$

(319) and $$\alpha = \pm \sqrt{\frac{C \omega}{2}} \sqrt{(R^2 + L^2 \omega^2)^{\frac{1}{2}} + L \omega}.$$

Substituting in (317) these values of the constants p and α, we find that the result is identical with (316) already obtained by a different method.

To Obtain the Current.

The current may be obtained from the potential equation by means of the relation $C \frac{d e}{d t} = - \frac{d i}{d x}$ [see (271)]. Differentiating (317) with respect to t and multiplying by C, we obtain

$$C \frac{d e}{d t} = C E \omega \epsilon^{p x} \cos (\omega t + \alpha x) = - \frac{d i}{d x}.$$

Integrating this result with respect to x, we get

$$i = - \frac{C E \omega \epsilon^{p x}}{p^2 + \alpha^2} \left\{ p \cos (\omega t + \alpha x) + \alpha \sin (\omega t + \alpha x) \right\}.$$

Transforming this into an equation containing the sine only, by means of formula (27), Chap. III., we obtain

$$(320) \quad i = -E \frac{\sqrt{C\omega}}{\sqrt[4]{R^2 + L^2\omega^2}} \epsilon^{px} \sin \left\{ \omega t + \alpha x + \tan^{-1} \sqrt{\frac{Im - L\omega}{Im + L\omega}} \right\}.$$

Here p and α represent the expressions (318) and (319).

Equation (320) may be written

$$i = -E \sqrt{\frac{C\omega}{Im}} \epsilon^{\pm px} \sin \left(\omega t \pm \alpha x + \tan^{-1} \frac{p}{\alpha} \right).$$

This equation gives the value of the current at any time at any point of a conductor containing distributed capacity and self-induction when subjected to an harmonic source of electromotive force. The discussion of this equation and the potential equation will be taken up in the following chapter.

CHAPTER XIII.

CIRCUITS CONTAINING DISTRIBUTED CAPACITY AND SELF INDUCTION.—DISCUSSION.

CONTENTS :—*Circuits with no self-induction.* Particular form of e and i equations. Nature of waves. Rate of propagation. Wave-length. Decreasing amplitude. Rate of decay with distance,—with time.

Circuits with self-induction. Phase difference. Rate of propagation. Diminishing amplitude. Rate of decay. Limitations of the telephone.

Wave-propagation in Closed Circuits.

Positive and negative waves travel around the circuit until they vanish. Resultant effect. Potential zero at middle point of the cable. Expression for potential simplified if the length of the cable is a multiple of a wave-length. Same results may be applied to the current equation.

IN order to ascertain the physical effects of distributed self-induction and capacity in a circuit, we will first discuss the analytical results obtained in the preceding chapter, as applied to a circuit in which the self-induction is neglected, and then, after investigating the nature of the wave-propagation, the wave-length, rate of propagation, and rate of decay, consider circuits containing distributed capacity and self-induction, noting the changes caused by the introduction of the self-induction.

Circuits with Distributed Capacity but no Self Induction.

When the effect of self-induction is not considered in the cable, we may put $L = 0$ in the current and potential equations (316) and (320) and reduce them to more simple forms, thus :

$$(321) \qquad e = E\epsilon^{\pm\sqrt{\frac{CR\omega}{2}}x} \sin\left\{\omega t \pm \sqrt{\frac{CR\omega}{2}}x\right\}.$$

$$(322) \qquad i = \pm E\sqrt{\frac{C\omega}{R}}\,\epsilon^{\pm\sqrt{\frac{CR\omega}{2}}x}$$

$$\sin\left\{\omega t \pm \sqrt{\frac{CR\omega}{2}}x + 45^\circ\right\}.$$

These results agree with those given by Mr. T. H. Blakesley in his book on "Alternating Currents of Electricity," page 60, second edition. They may be directly obtained from the differential equation (272), and upon differentiation will be found to satisfy it.

Nature of the Wave-propagation.—Equation (321) shows that at any point of the conductor the potential varies harmonically with the time. At the origin where $x = 0$, the potential is always equal to $e = E \sin \omega t$, its maximum value being E; but as we proceed from the origin the potential becomes less, being equal to $E\epsilon^{-\sqrt{\frac{CR\omega}{2}}x}$, an expression which decreases as x increases. The double sign is retained in the exponent in the equation, since it represents two waves, one going in the positive and the other in the negative direction from the origin, which is the source of alternating potential. The maximum value of the potential at any point of the cable may be represented as in Fig. 44 by an ordinate to the logarithmic curve;

thus, $\overline{OA}$ represents the maximum value of the harmonically varying potential at the origin, and $\overline{BC}$ its maximum value at a distance x from the origin. The distance $\overline{OB}$ is taken as that in which the logarithmic curve has decreased to $\frac{1}{\epsilon}$ of its original value, $\overline{OA}$. The distance $\overline{OD}$ is a quarter wave-length, and $\overline{OE}$ a half wave-length. It will

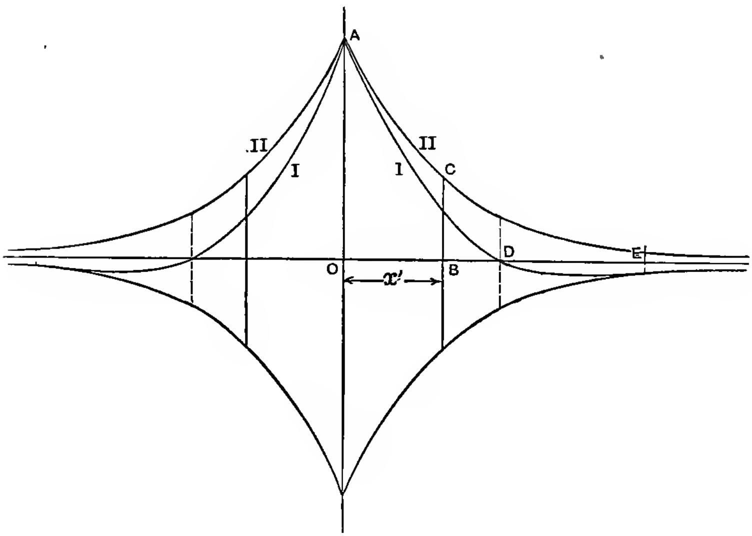


FIG. 44.—CURVE I.—INSTANTANEOUS WAVE IN INFINITE CABLE.
CURVE II.—LOGARITHMIC DECREASE IN AMPLITUDE.

presently be shown that the amplitude decreases to almost $\frac{1}{500}$ of its original value in one complete wave-length.

One of the most striking results shown by these formulæ is that the current always precedes the potential by one eighth of a period, and this difference in phase is not altered by any change in the resistance or capacity of the cable.

Rate of Propagation.—Curve I. (Fig. 44) represents an instantaneous position of the potential wave travelling along the wire in each direction from the origin. The distance

along the cable from one maximum potential to the next, or in other words the length of one complete wave, may be found by equating to 2π the angle containing x in the equation. This gives

$$\sqrt{\frac{C R \omega}{2}}x = 2\pi,$$

or
$$x = \lambda = 2\pi\sqrt{\frac{2}{C R \omega}} = 2\sqrt{\frac{\pi}{C R n}},$$

where λ denotes the length of one complete wave, and $n = \frac{\omega}{2\pi}$ the frequency of alternation.

The time of one complete period is represented by $T = \frac{1}{n}$, and in that time the wave advances a distance λ, equal to one wave-length. The rate at which the wave advances is found by dividing the wave-length by the time taken in advancing that distance; thus,

$$\text{Rate of propagation} = \frac{\lambda}{T} = 2\sqrt{\frac{n\pi}{CR}} = \sqrt{\frac{2\omega}{CR}}.$$

It is seen that the rate of propagation not only depends upon the character of the cable, but likewise varies as the square root of the frequency of alternation. A wave with a frequency of 400 will travel twice as fast as one with a frequency of 100 alternations per second.

Decreasing Amplitude.—The frequency affects not only the rate of propagation, but also has a marked influence upon the rate at which the amplitude of the waves decreases with the distance. The distance, which we will call x', at which the amplitude of the wave will have $\frac{1}{\epsilon}$ of its original

value, is the reciprocal of the coefficient of x in the exponent, thus:

$$x' = \sqrt{\frac{2}{C R \omega}}.$$

This distance, at which the wave decays to $\frac{1}{\epsilon}$ of its value at the origin, varies inversely as the square root of the frequency; this means that a wave with a frequency of 100 alternations per second can go twice as far as one with a frequency of 400 and experience the same decay in amplitude. Comparing this value of x' with that of a wave-length λ, we may write

$$x' = \frac{\lambda}{2\pi}.$$

The amplitude of a wave, therefore, decreases to $\frac{1}{\epsilon^{2\pi}} = .00187$ of its value in advancing a distance equal to one wave-length.

In order to find the time, t', in which the amplitude decays to $\frac{1}{\epsilon}$ of its value, we must divide this distance, x', by the rate of propagation, thus:

$$t' = \frac{\frac{\lambda}{2\pi}}{\frac{\lambda}{T}} = \frac{T}{2\pi} = \frac{1}{\omega}.$$

We see that the time of decay varies inversely as the frequency; thus a wave with a frequency of 400 alternations per second will decrease a certain amount in one fourth

the time a wave with a frequency of 100 is decreasing the same amount.

Circuits with Distributed Capacity and Self Induction.

It will be noticed that the introduction of self-induction into the cable, in the manner before described, does not materially alter the character of the wave propagation. This is evident from the equations (316) and (320). At every point of the conductor the potential, or current, varies harmonically with the time, and, as before, the amplitude of the wave decreases with a logarithmic decrement as it proceeds from the origin.

Phase Difference.—One effect of the self-induction is to change the angle of advance of the current ahead of the potential. This angular difference is no longer a constant angle of 45°, as formerly, but is a function depending upon R, L, and ω, viz.,

$$\tan\theta = \sqrt{\frac{Im - L\omega}{Im + L\omega}} = \frac{p}{\alpha}. \tag{323}$$

When $L = 0$, $\tan\theta = 1$, and $\theta = 45°$. As L increases from zero to infinity the expression changes from unity to zero, and the angle θ, consequently, from 45° to 0°. The effect of the self-induction is, therefore, to decrease the phase difference between the current and potential. This difference of phase also becomes less with an increase in the frequency.

Rate of Propagation.—The distance along the cable between two maxima is found by equating the angle containing x in equation (316) to 2π and solving for x, thus:

$$x = \lambda_1 = \frac{2\pi}{\alpha} = \frac{2\pi\sqrt{2}}{\sqrt{C\omega}\sqrt{Im + L\omega}},$$

where λ_1 denotes the wave-length. Subscripts are here used to denote circuits with self-induction. The time occupied in travelling this distance is $T = \frac{2\pi}{\omega}$; hence,

$$\text{Rate of propagation} = \frac{\lambda_1}{T} = \frac{\omega}{\alpha} = \sqrt{\frac{2\,\omega}{C\{Im + L\,\omega\}}}.$$

It is to be noticed that the wave-length and the rate of propagation are each less than that found for circuits containing no self-induction. When $L = 0$, these expressions just found reduce to those previously given.

Another point to be noticed is that a change in frequency will have a greater effect in altering the wave-length but not so great an effect in changing the rate of propagation as in the case of a circuit with no self-induction. Two waves of different periods will, therefore, go more slowly but with less difference in their rates of propagation than with no self-induction. As before, the wave of higher frequency will have the shorter wave-length and advance the faster.

Decreasing Amplitude.—As before, the amplitude of the harmonic wave has a logarithmic decrement, decreasing with the distance from the origin. The distance at which the amplitude has $\frac{1}{\epsilon}$ of its original value is the reciprocal of the coefficient of the exponent, thus:

$$x_1' = \frac{1}{p} = \frac{\sqrt{\frac{2}{C\,\omega}}}{\sqrt{Im - L\,\omega}}.$$

This is larger on account of the self-induction. The substitution of $L = 0$ reduces it to the value of x' found before. An increase in frequency will cause this value to

decrease, and the decay in a certain distance to increase; but the frequency has not so great an influence upon this decay as it has in a circuit with no self-induction.

Comparing with the values found for λ_1 and θ, we see that

$$x_1' = \frac{\lambda_1}{2\pi \tan\theta}.$$

In order to find the time, t_1', in which the amplitude decays to $\frac{1}{\epsilon}$ of its original value, we divide the distance x_1' by the rate of propagation, thus:

$$t_1' = \frac{\dfrac{\lambda_1}{2\pi \tan\theta}}{\dfrac{\lambda_1}{T}} = \frac{T}{2\pi \tan\theta} = \frac{1}{\omega \tan\theta}.$$

The self-induction causes $\tan\theta$ to have a value less than unity, thus increasing the time for a certain decay,—that is, decreasing the rate. An increase in the frequency causes θ to become smaller. The exact effect upon the rate of decay caused by a variation in frequency in a circuit with self-induction depends upon the constants of the circuit. The wave of higher frequency will always decay the more rapidly, but with self-induction in the circuit there is less difference in the rates of decay of waves of different periods than there is in circuits without it.

Limitations of the Telephone.—The effects of distributed capacity in a conductor upon the wave-propagation have been given, and the way in which these effects are altered by the introduction of self-induction or by a change in frequency. A consideration of the results with reference to telephone circuits is valuable inasmuch as it is just such effects that cause the limitations to telephony. In all cases

the waves of higher frequency travel the faster, and so the several harmonic components of a complex tone are constantly changing in their relative phases. The waves of higher frequency are likewise subject to the more rapid decay, and so when the several components are recombined the resultant tone may be materially altered from the original complex tone. These effects may be modified by the presence of self-induction, but in all cases they will be present to a certain extent, thus defining the limits of the use of the telephone.

Wave-propagation in Closed Circuits.

Let us consider that a dynamo giving an harmonic alternating E. M. F. is inserted at some point of a cable, such as that described, which forms a continuous closed circuit. There will be a forward wave of positive potential starting from one pole of the machine which will travel around and around the circuit, continually diminishing in amplitude, until it finally vanishes. At the same time a backward wave of negative potential will start from the other pole of the machine and travel around in the opposite direction until it too vanishes. The potential at any particular point of the circuit is thus the sum of the potentials due to all the positive and negative waves. Let l denote the length of the cable. When the first positive wave reaches a point at a distance x from the pole of the dynamo, the potential due to this forward wave is

$$e_{F_1} = E\epsilon^{-px} \sin(\omega t - \alpha x),$$

where p and α have the values given in (318) and (319). When the wave has travelled completely around the circuit and comes to this point a second time, its value is

$$e_{F_2} = E\epsilon^{-px-pl} \sin(\omega t - \alpha x - \alpha l),$$

which may be obtained by substituting $x+l$ for x in the above. After going n times around, it has become

$$e_{F_n} = E\epsilon^{-px-pln}\sin(\omega t - \alpha x - \alpha l n).$$

The resultant of all these forward waves at any one point may be written, therefore,

$$(324)\quad e_F = e_{F_1} + e_{F_2} + \dots + e_{F_\infty}$$

$$= E\epsilon^{-px}\sum_{n=0}^{n=\infty}\epsilon^{-pln}\sin(\omega t - \alpha x - \alpha l n).$$

When we consider all the backward waves, they may be represented by a similar expression in which E has the negative sign. When the first backward wave has travelled around the circuit so as to be a positive distance x from the origin, it has travelled a distance $l - x$; this distance from the origin in a negative direction we will call x'. We may therefore write for the first backward wave

$$e_{B_1} = -E\epsilon^{-px'}\sin(\omega t - \alpha x'),$$

and for the sum of all the backward waves

$$(325)\quad e_B = -E\epsilon^{-px'}\sum_{n=0}^{n=\infty}\epsilon^{-pln}\sin(\omega t - \alpha x' - \alpha l n).$$

These expressions (324) and (325) may be simplified since we may put

$$(326)\quad \sin(\omega t - \alpha x - \alpha l n) = \sin(\omega t - \alpha x)\cos\alpha l n$$

$$- \cos(\omega t - \alpha x)\sin\alpha l n.$$

Substituting (326) in (324), we have

$$(327) \quad e_F = E\epsilon^{-px} \sin(\omega t - \alpha x) \sum_{n=0}^{n=\infty} \epsilon^{-pln} \cos \alpha l n$$

$$- E\epsilon^{-px} \cos(\omega t - \alpha x) \sum_{n=0}^{n=\infty} \epsilon^{-pln} \sin \alpha l n.$$

Similarly we may reduce (325) to

$$(328) \quad e_B = - E\epsilon^{-px'} \sin(\omega t - \alpha x') \sum_{n=0}^{n=\infty} \epsilon^{-pln} \cos \alpha l n$$

$$+ E\epsilon^{-px'} \cos(\omega t - \alpha x') \sum_{n=0}^{n=\infty} \epsilon^{-pln} \sin \alpha l n.$$

The resulting potential at any point due to the forward and backward waves is the sum of E_F and E_B. Writing $l - x$ for x' in (328) and adding to (327), we obtain

$$(329) \quad e = E\sum_{n=0}^{n=\infty} \epsilon^{-pln} \cos \alpha l n \left\{ \epsilon^{-px} \sin(\omega t - \alpha x) \right.$$

$$\left. - \epsilon^{+px-pl} \sin(\omega t + \alpha x - \alpha l) \right\} + E\sum_{n=0}^{n=\infty} \epsilon^{-pln} \sin \alpha l n$$

$$\left\{ \epsilon^{+px-pl} \cos(\omega t + \alpha x - \alpha l) - \epsilon^{-px} \cos(\omega t - \alpha x) \right\}.$$

By means of the exponential values of the sine and cosine, the values of the two summations expressed in (329) are found * to be

$$(330) \quad \sum_{n=0}^{n=\infty} \epsilon^{-pln} \sin \alpha l n = \frac{\epsilon^{pl} \sin \alpha l}{1 - 2\epsilon^{pl} \cos \alpha l + \epsilon^{2pl}},$$

---

* Equations (330) and (331) may be verified thus: For brevity put $pl = h$, and $\alpha l = k$. Writing the exponential value of the sine [see equation (109), Chap. VII.], we have

and

$$(331)\qquad \sum_{n\ 0}^{n=\infty} \epsilon^{-pln} \cos \alpha l n = \frac{\epsilon^{2pl} - \epsilon^{pl}\cos \alpha l}{1 - 2\epsilon^{pl}\cos \alpha l + \epsilon^{2pl}}.$$

$$\sum_{n=0}^{n=\infty} \epsilon^{-hn} \sin kn = \sum_{n=0}^{n=\infty} \epsilon^{-hn} \cdot \frac{\epsilon^{jkn} - \epsilon^{-jkn}}{2j}$$

$$= \frac{1}{2j}\sum_{n=0}^{n=\infty} \epsilon^{jkn-hn} - \frac{1}{2j}\sum_{n=0}^{n=\infty} \epsilon^{-(jkn+hn)}.$$

Thus we have the given series equivalent to the difference of two infinite decreasing geometrical series. The sum of such a series is known to be equal to the first term divided by unity minus the common ratio, i.e., $s = \frac{a}{1-r}$, where s denotes the sum, a the first term, and r the common ratio. Applying this formula, the sum of the first series is $\frac{1}{2j} \cdot \frac{1}{1 - \epsilon^{jk-h}}$; and of the second $\frac{1}{2j} \cdot \frac{1}{1 - \epsilon^{-(jk+h)}}$. Hence

$$\sum_{n=0}^{n=\infty} \epsilon^{-hn} \sin kn = \frac{1}{2j}\left\{\frac{1}{1 - \epsilon^{jk-h}} - \frac{1}{1 - \epsilon^{-(jk+h)}}\right\}.$$

Multiplying both numerator and denominator by ϵ^{h} and reducing the terms in the brackets to a common denominator after factoring out the factor ϵ^{h}, we have

$$\sum_{n=0}^{n=\infty} \epsilon^{-hn} \sin kn = \frac{\epsilon^{h}}{2j}\left\{\frac{\epsilon^{jk} - \epsilon^{-jk}}{\epsilon^{2h} - \epsilon^{h}\left(\epsilon^{jk} + \epsilon^{-jk}\right) + 1}\right\}.$$

Replacing the exponential values of the sine and cosine, we have

$$\sum_{n=0}^{n=\infty} \epsilon^{-hn} \sin kn = \frac{\epsilon^{h}\sin k}{\epsilon^{2h} - 2\epsilon^{h}\cos k + 1} = \frac{\epsilon^{pl}\sin \alpha l}{\epsilon^{2pl} - 2\epsilon^{pl}\cos \alpha l + 1}.$$

In a similar manner we may verify equation (331), thus:

$$\sum_{n=0}^{n=\infty} \epsilon^{-hn} \cos kn = \frac{1}{2}\sum_{n=0}^{n=\infty} \epsilon^{jkn-hn} + \frac{1}{2}\sum_{n=0}^{n=\infty} \epsilon^{-(jkn+hn)}$$

$$= \frac{1}{2}\left\{\frac{1}{1 - \epsilon^{jk-h}} + \frac{1}{1 - \epsilon^{-(jk+h)}}\right\}$$

Let POP (Fig. 45) represent the cable which is supposed to form a closed circuit, the ends at P being joined

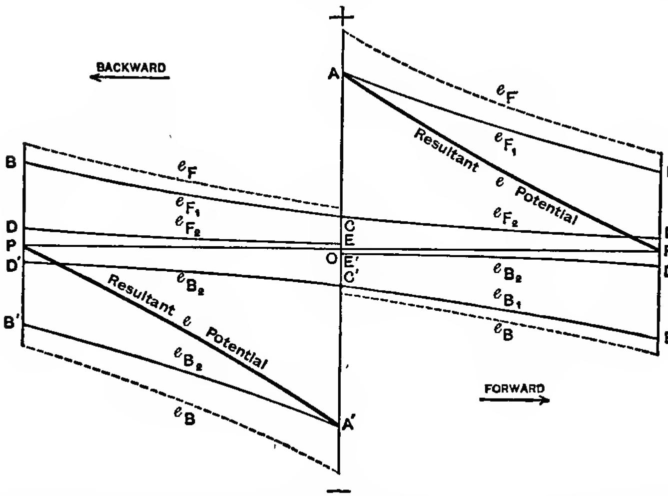


Fig. 45.—Forward and Backward Waves, and Resultant Potential, in a Closed Conductor.

together. The maximum value of the potential at the positive pole of the dynamo is represented by $\overline{OA}$. As we go from A, this decreases along the logarithmic curve $ABCDE$, until it finally vanishes altogether. Similarly, a

---

$$= \frac{\epsilon^h}{2}\left\{\frac{2\epsilon^h - \epsilon^{jk} - \epsilon^{-jk}}{\epsilon^{2h} - \epsilon^h(\epsilon^{jk} + \epsilon^{-jk}) + 1}\right\}$$

$$= \frac{\epsilon^h}{2}\left\{\frac{2\epsilon^h - 2\cos k}{\epsilon^{2h} - 2\epsilon^h \cos k + 1}\right\}.$$

Therefore

$$\sum_{n=0}^{n=\infty} \epsilon^{-pln} \cos \alpha l n = \frac{\epsilon^{2pl} - \epsilon^{pl}\cos \alpha l}{\epsilon^{2pl} - 2\epsilon^{pl}\cos \alpha l + 1}.$$

Q.E.D.

backward wave coming from the negative pole decreases along the curve $A' B' C' D' E'$. At the point P, half-way between the poles of the dynamo, the middle point of the cable, it is evident that the potential remains continually zero, for, at the point P, the distance x is $\frac{l}{2}$, which reduces equation (329) to zero.

If the length of the cable happens to be some multiple of a wave-length, the expression for the potential takes a simpler form. In this case each successive forward wave travels around the circuit in the same phase as the first, and all these forward waves may, therefore, be added together algebraically. The maximum resultant potential at any point will be the sum of the maxima of the separate waves.

In Fig. 45, e_{F_1}, e_{F_2} represent successive forward waves, and e_{B_1} and e_{B_2} the corresponding backward waves. In the case where the length of the cable is a multiple of the wave-length, the sum of the maxima of all the forward and of all the backward waves is represented by the dotted lines e_F and e_B, respectively. The solid line e, the sum of e_F and e_B, represents the resultant maximum potential along the conductor.

We have seen that the wave-length is $\lambda = \frac{2\pi}{\alpha}$. The length of the cable is a multiple of the wave-length $l = \kappa\lambda = \frac{2\pi\kappa}{\alpha}$, and $\alpha l = 2\pi\kappa$, where κ is a positive integer. This value reduces (330) to zero, since $\sin 2\kappa\pi = 0$, and reduces (331) to

$$\frac{\epsilon^{2pl} - \epsilon^{pl}}{(1 - \epsilon^{pl})^2} = \frac{\epsilon^{pl}}{\epsilon^{pl} - 1}.$$

These values cause the second term in (329) to vanish, and the whole becomes

$$(332) \qquad e = E\frac{\epsilon^{px} - \epsilon^{pl-px}}{1 - \epsilon^{pl}} \sin(\omega t - \alpha x),$$

which expresses the resultant potential, represented by the solid line e in Fig. 45, at any point of the cable, provided its length is some multiple of a wave-length. When $x = 0$, this reduces to $e = E \sin \omega t$, the expression for the potential at the terminals of the dynamo. When $x = \frac{l}{2}$, the expression vanishes, showing that the potential is constantly zero at the middle point of the conductor.

This last simplification was made possible by considering the length of the cable to be a multiple of the wave-length; otherwise the algebraic addition of the maxima of the several waves would not be possible, since they would differ in phase. The construction of the resultant curves would not be so simple, but the nature of the results would not be materially modified.

The phenomena in connection with the flow of current are similar to those just discussed relating to the propagation of potential, and are obtained in the same manner from the current equation.

PART II.

GRAPHICAL TREATMENT.

CHAPTER XIV.

INTRODUCTORY TO PART II. AND TO CIRCUITS CONTAINING RESISTANCE AND SELF INDUCTION.

CONTENTS:—Introductory. Analytical solutions of Part I. for simple circuits extended to compound circuits by graphical method. Arrangement of Part II. Graphical representation of simple harmonic E. M. F.'s. Graphical representation of the sum of simple harmonic E. M. F.'s of same period. Triangle of E. M. F.'s for a single circuit containing resistance and self-induction. Impressed E. M. F. Effective E. M. F. Counter E. M. F. of self-induction. Direction shown from differential equation. Graphical representation. Methods to be used and symbols adopted in the graphical treatment of problems. First method (the one used throughout this book), employing E. M. F. necessary to overcome self-induction. Second method, employing E. M. F. of self-induction. System of lettering and conventions adopted in graphical construction.

THE analytical solutions derived in Part I. apply merely to a single circuit having resistance, self-induction, and capacity in series. The problems which arise in case there is not simply a single circuit but a complicated network of conductors might be treated analytically, though the process would be exceedingly laborious and the results too cumbersome to handle. Fortunately, however, by making use of the analytical solutions already given in Part I., and extending them by graphical methods, we are enabled to solve problems with compound circuits which offer con-

siderable difficulty to analytical investigation. These graphical methods are most easily and advantageously adapted to problems in which we deal with an harmonic impressed E. M. F.

The object of this Part is to show how to solve by graphical methods any problems arising with any combination of series and parallel circuits, in any branch of which there may be an harmonic impressed E. M. F.

The plan to be followed is similar to that adopted in the first Part. First are considered various compound circuits which contain resistance and self-induction only, and then circuits containing resistance and capacity only, and finally circuits containing all three, resistance, self-induction, and capacity. The problems to be considered in each case are similar, first a series circuit, then a divided circuit with two branches and with any number of branches, then any combination of series and parallel circuits.

Before giving the solutions of these problems, the way in which this graphical method corresponds to and is a substitute for the analytical method, and the manner in which it is to be used, will be explained.

Graphical Representation of a Simple Harmonic Electromotive Force.

An harmonic impressed electromotive force is represented by the equation

$$e = E \sin \omega t,$$

as was explained in Chap. II. on harmonic functions. The plot of this equation, in which t is the independent and e the dependent variable, gives the sine-curve represented in Fig. 46. A diagrammatic method of representing this harmonic E. M. F. is seen in the same figure. The line $\overline{OA}$ is supposed to revolve in the counter-clockwise direction

about the point O with uniform angular velocity. Its projection $\overline{OP}$ at any moment corresponds to the ordinate $O'P'$ of the sine-curve. If the circle be moved horizontally with a constant velocity, the projection $\overline{OP}$ would trace a sine-curve the ordinates of which represent the value of the impressed E. M. F. at any instant. Diagrammatically we may represent the impressed E. M. F. by the line $\overline{OA}$

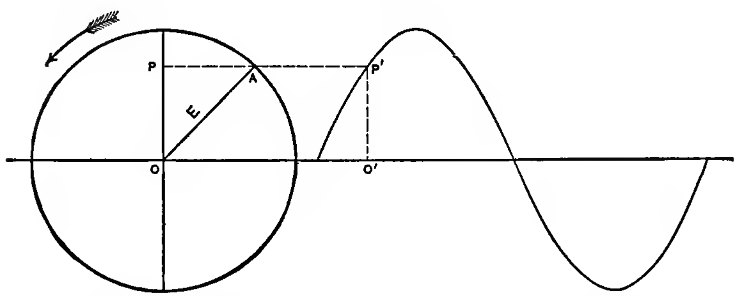


FIG. 46.—GRAPHICAL REPRESENTATION OF A SIMPLE HARMONIC ELECTROMOTIVE FORCE.

alone, which is equal in length to its maximum value, E. In this sense, then, we may represent harmonic E. M. F.'s by lines in the graphical constructions which follow.

GRAPHICAL REPRESENTATION OF THE SUM OF SIMPLE HARMONIC ELECTROMOTIVE FORCES HAVING THE SAME PERIOD.

If an E. M. F. is the sum of two simple harmonic E. M. F.'s of the same period, it may be represented by the equation

$$(333) \qquad e = E_1 \sin \omega t + E_2 \sin (\omega t + \theta).$$

It can easily be shown analytically that this sum is a simple harmonic E. M. F., differing in phase and amplitude from each of the two components, and having the same

period; for, upon expanding $\sin\ (\omega t + \theta)$, the equation becomes

$$e = (E_1 + E_2 \cos\theta) \sin \omega t + E_2 \sin\theta \cos\omega t.$$

This may be transformed by means of the trigonometric formula (27), Part I., to

$$(334)\quad e = \sqrt{E_1^2 + E_2^2 + 2 E_1 E_2 \cos\theta}\ \sin\left\{\omega t + \tan^{-1}\frac{E_2 \sin\theta}{E_1 + E_2 \cos\theta}\right\}.$$

This equation represents a simple harmonic E. M. F., since it is of the form

$$e = E \sin (\omega t + \phi),$$

in which E and ϕ are constant quantities. Moreover, this equation shows that the diagonal of the parallelogram formed by the two component lines which represent the two component terms of equation (333) is the line which graphically represents equation (334), and is therefore the sum of the two components.

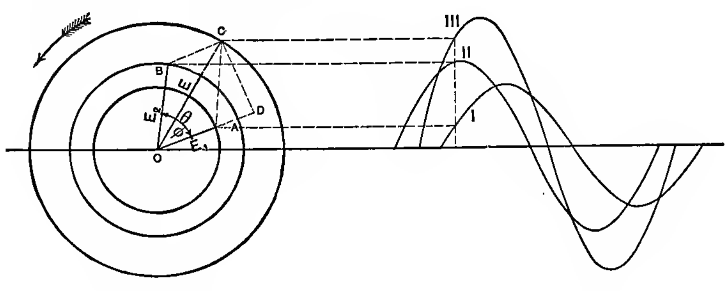


FIG. 47.—RESULTANT OF TWO HARMONIC ELECTROMOTIVE FORCES.

In Fig. 47, curve I., generated by the line $\overline{OA}$, represents the first term of equation (333). Curve II., generated by $\overline{OB}$, represents the second term. Curve III. is the sum of

curves I. and II., and is generated by the diagonal $\overline{O\,C}$ of the parallelogram formed upon the two components $\overline{O\,A}$ and $\overline{O\,B}$. That curve III., the geometrical sum, represents equation (334), the analytical sum, is seen by the fact that the analytical relations, as shown by the equation, agree with those readily obtained from the geometry of the figure. Thus, from the equation, the amplitude of the resultant harmonic function must be

$$E = \sqrt{E_1^2 + E_2^2 + 2\,E_1\,E_2 \cos\theta}.$$

But from the geometry of the figure this same relation is evident, for $\overline{O\,A} = E_1$, $\overline{O\,B} = E_2$, and $A\;O\;B = \theta$.

Again, from the equation the resultant E. M. F. differs in phase from E_1 by an angle

$$\phi = \tan^{-1} \frac{E_2 \sin\theta}{E_1 + E_2 \cos\theta}.$$

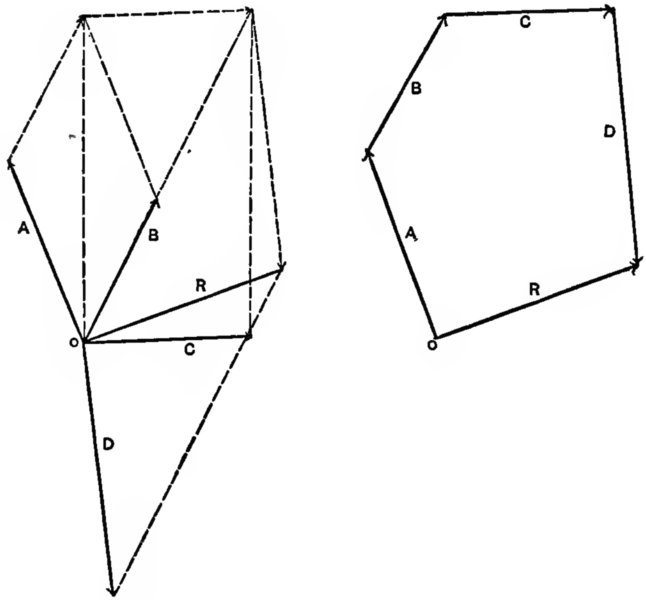
From the figure we see

$$\phi = C\;O\;A = \tan^{-1} \frac{\overline{C\,D}}{\overline{D\,O}} = \tan^{-1} \frac{E_2 \sin\theta}{E_1 + E_2 \cos\theta}.$$

This agreement of the analytical and graphical relations establishes the correctness of the construction, and we can, therefore, conclude that the sum of any two sine-curves of the same period represented by two lines revolving about a common centre is also a sine-curve of the same period represented by the diagonal of the parallelogram formed on the two component lines.

When the component E. M. F.'s are more than two in number the sum is represented by the vector, which is the geometrical resultant of all the component vectors. This evidently follows from the preceding, since any two compo

nents are equivalent to a single E. M. F., and this combined with a third and fourth component gives the geometrical resultant as the sum of all the components. Thus, in Fig.



FIGS. 48 AND 49.—ADDITION OF HARMONIC ELECTROMOTIVE FORCES.

48, we have a number of vectors A, B, C, D, each representing one component E. M. F. and drawn from the same origin O. The sum is found in the usual manner by constructing a parallelogram on any two and then combining the resultant with a third, and so on until all the components are reduced to a single resultant vector R. This process is equivalent to that indicated in Fig. 49, where the vector A is first drawn from the origin O, then B from the extremity of A, C from the extremity of B, and so on until all the lines are drawn. The resultant or geometrical sum is then the vector, R, drawn from the origin to the last point found, thus completing a closed polygon.

Triangle of Electromotive Forces for a Single Circuit containing Resistance and Self-induction.

In Chapter III., in which circuits containing resistance and self-induction were analytically treated, it was shown that if a circuit contains an harmonic impressed E. M. F.,

$$e = E \sin \omega t,$$

the value of the current is also harmonic and is

$$(335) \qquad i = \frac{E}{\sqrt{R^2 + L^2 \omega^2}} \sin \left(\omega t - \tan^{-1} \frac{L \omega}{R} \right).$$

This current equation was derived from the differential equation of electromotive forces

$$e = R i + L \frac{d i}{d t},$$

in which e is the instantaneous value of the impressed E. M. F. of the source, and $R i$ that part, usually called the effective E. M. F., necessary to overcome the ohmic resistance, and $L \frac{d i}{d t}$ that part necessary to overcome the counter E. M. F. of self-induction.

In Fig. 50 let the vector $\overline{O A}$ represent the harmonic impressed E. M. F. Then, by equation (335), we know that the current is represented by a vector $\overline{O B}$ lagging behind $\overline{O A}$ by an angle θ whose tangent is $\frac{L \omega}{R}$.

The effective E. M. F. must be represented by a vector $\overline{O C}$, equal to RI, in the same direction as the current, and equal to the current vector, $\overline{O B}$, multiplied by R. The counter E. M. F. of self-induction, $L \frac{d i}{d t}$, is at right angles

to the current and must therefore be represented by the vector $\overline{CA}$ perpendicular to $\overline{OB}$. It can be shown to be

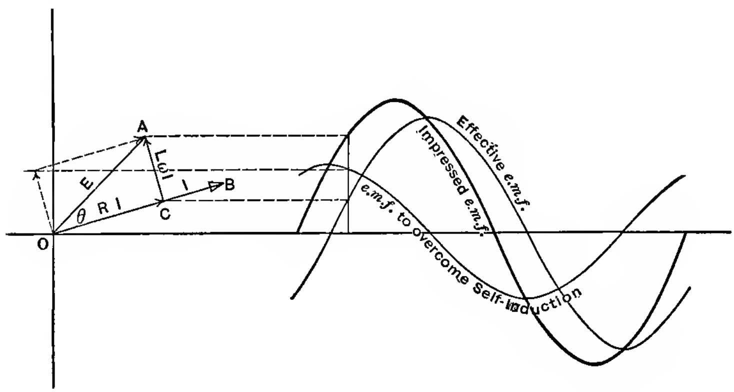


FIG. 50.—TRIANGLE OF ELECTROMOTIVE FORCES. FIRST METHOD—THE ONE USED THROUGHOUT THIS BOOK—EMPLOYING E. M. F. TO OVERCOME SELF-INDUCTION.

at right angles to the current, as follows. Equation (335) may be written thus:

$$i = I \sin (\omega t - \theta).$$

By differentiation,

$$\frac{di}{dt} = \omega I \cos (\omega t - \theta). \tag{336}$$

Multiplying this equation by L and writing in terms of the sine, we have

$$L \frac{di}{dt} = L \omega I \sin (\omega t - \theta + 90°). \tag{337}$$

By this equation it is seen that the E. M. F., $L \frac{di}{dt}$, *necessary to overcome* that of self-induction is represented by a vector $\overline{CA}$, whose length is $L \omega I$, ninety degrees in advance of the

current. The E. M. F. *of* self-induction is equal and opposite to that which is *necessary to overcome* it, and is consequently ninety degrees behind the current, represented by the vector $\overline{AC}$.

That the foregoing construction represents the case and fulfils the analytical conditions expressed by the current equation (335) may be shown again by a further comparison of the geometrical with the analytical relations. Thus in Fig. 50 or 51,

$$\tan AOC = \frac{\overline{AC}}{\overline{OC}} = \frac{L\omega I}{RI} = \frac{L\omega}{R} = \tan\,\theta.$$

This is seen to correspond to the angle of lag in equation (335). Also the impressed E. M. F. $\overline{OA}$, being the hypotenuse of the triangle OAC, is equal to the square root of the sum of the squares of the two sides, and therefore

$$\overline{OA} = \sqrt{\overline{OC}^2 + \overline{CA}^2};$$

that is, $E = \sqrt{R^2 I^2 + L^2 \omega^2 I^2} = I\sqrt{R^2 + L^2 \omega^2}$.

and $$I = \frac{E}{\sqrt{R^2 + L^2 \omega^2}}.$$

This is seen to correspond to the maximum value of the current given in equation (335).

Method to be Used and Symbols Adopted in the Graphical Treatment of Problems.

In the graphical treatment of circuits with resistance and self-induction there are two methods, each equally correct, for obtaining the same results depending upon whether we consider the E. M. F. of self-induction or the equal and opposite E. M. F. necessary to overcome it.

The method in which the E. M. F. necessary to overcome the self-induction is used is shown in Fig. 50 and has been fully discussed. In this method of construction, the impressed E. M. F. is regarded as made up of the sum of two components, one the effective E. M. F. in the direction of the current, and the other that *necessary to overcome* self-induction ninety degrees *ahead* of the current.

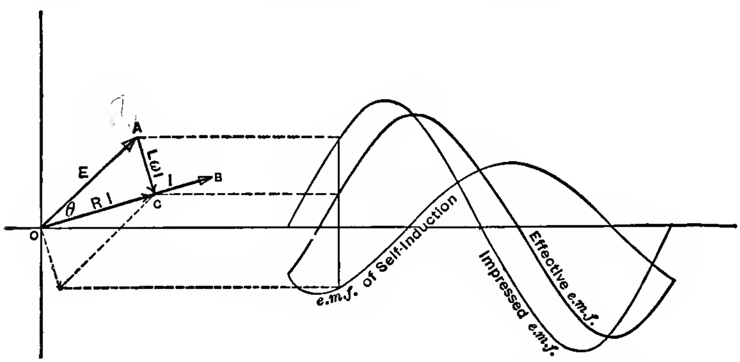


FIG. 51.—TRIANGLE OF ELECTROMOTIVE FORCES. SECOND METHOD, EMPLOYING E. M. F. OF SELF-INDUCTION.

The method in which the E. M. F. of self-induction is used is shown in Fig. 51. The point of difference is that the line $\overline{AC}$ represents the E. M. F. *of* self-induction instead of the E. M. F. *necessary to overcome it*, and is ninety degrees *behind* the current instead of *ahead* of it.

In this method of drawing, the effective E. M. F. which drives the current is regarded as the resultant of the two other E. M. F.'s in the circuit, viz., the impressed E. M. F. and that of self-induction.

Either of these methods, if carried throughout the whole drawing, is correct and finally brings the same results; but unless one method is adopted and carried throughout, there is apt to be confusion. In all cases the method used in this book is the first one, namely, that which considers

the E. M. F. as that *necessary to overcome* self-induction, as illustrated in Fig. 50.

In order that the diagrams may be readily understood, the arrangement and system of lettering adopted will be explained. In all cases the *positive* direction is *counter-clockwise*, and the diagram is supposed to revolve counter-clockwise around the centre O. Lines will be designated by letters in small capitals placed at their extremities. The letters therefore designate points and will be used alphabetically, beginning with A, in the order in which the points are determined. Thus in Fig. 54 the line $\overline{OA}$ is first drawn, then the points B, C, D, etc., are determined, and the lines $\overline{OB}$, $\overline{OC}$, $\overline{BC}$, $\overline{CD}$, etc., drawn in order. All revolve counter-clockwise about O.

The direction of lines representing E. M. F. or current will be indicated by arrows, and, where possible, these arrows will be placed so as to show where the lines terminate. In order that lines representing current and E. M. F. may be distinguished, the arrows for current will have a closed head, as in the case of the line $\overline{OA}$, Fig. 54, and the arrows for E. M. F. will have an open head, as on the line $\overline{BC}$. Dotted lines are needed only for the construction of the figure or to make clear some point that would otherwise be ambiguous or doubtful. When a number of lines terminate at one point and are each directed toward the point, it has been found convenient to avoid the confusion of the many arrows coming thus together by omitting the arrows and drawing a small circle at the point, as at G, Fig. 54.

CHAPTER XV.

PROBLEMS WITH CIRCUITS CONTAINING RESISTANCE AND SELF INDUCTION. SERIES CIRCUITS AND DIVIDED CIRCUITS.

Problem I. Effects of the Variation of the Constants R and L in a Series Circuit.

Before taking up the problems proper which arise in connection with the investigation of circuits containing resistance and self-induction, it will be well to first consider the changes which occur when the resistance is varied and the coefficient of self-induction kept constant, and those

which occur when the self-induction is varied and the resistance kept constant. The limiting cases, when the resistance or the self-induction approaches zero or infinity, will be shown, so that the following problems may be applied to such limiting cases without the confusion which might otherwise arise.

Resistance Varied.

Let us suppose that the ohmic resistance is varied in a circuit in which the self-induction is kept constant.

Let OAC, Fig. 52, represent the triangle of E. M. F.'s

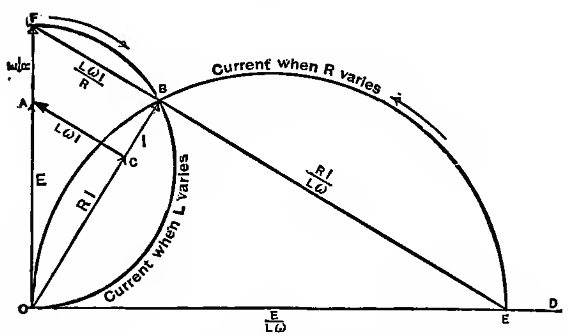


FIG. 52.—VARIATION OF RESISTANCE AND SELF-INDUCTION IN A SERIES CIRCUIT. PROBLEM I.

for the circuit when the resistance is R. Divide $\overline{OC}$ by R to obtain the current I equal to $\overline{OB}$. Draw the line $\overline{OD}$ of indefinite length perpendicular to the E. M. F. $\overline{OA}$ in the direction of lag. The angle DOC, being the complement of AOC, is therefore $\tan^{-1}\frac{R}{L\omega}$. Draw $\overline{BE}$ perpendicular to $\overline{OB}$ and let it meet the line $\overline{OD}$ in E. Then in the right triangle OBE the side $\overline{BE}$ equals $\frac{RI}{L\omega}$; for OB equals I, and $\tan EOB$ equals $\frac{R}{L\omega}$.

It follows that the hypotenuse, $\overline{OE}$, of this triangle is equal to $\frac{E}{L\omega}$, and is, therefore, a constant entirely independent of any variation in the current I, or resistance R. Taking the square root of the sum of the squares of the sides $\overline{OB}$ and $\overline{BE}$, we obtain

$$\overline{OE} = \sqrt{\overline{OB}^2 + \overline{BE}^2} = I\sqrt{1 + \frac{R^2}{L^2\omega^2}}.$$

From equation (29) we have

$$I = \frac{E}{\sqrt{R^2 + L^2\omega^2}};$$

Therefore

$$\overline{OE} = \frac{E}{\sqrt{R^2 + L^2\omega^2}}\sqrt{1 + \frac{R^2}{L^2\omega^2}} = \frac{E}{L\omega}.$$

Now, since the side $\overline{OB}$ of the right triangle OBE always represents the current I, and the hypotenuse $\overline{OE}$ is independent of the current or the resistance, it follows that the current is always represented by a vector $\overline{OB}$ inscribed in the semi-circle OBE, for any possible variation in the resistance. The arrow shows the direction of change as R increases.

In the particular cases when R is infinite or zero we see clearly by this figure the limiting values of the current. When R is infinite the current is evidently zero. When R approaches zero (or, what is approximately the same thing, becomes very small compared with the self-induction) $\overline{OB}$ approaches $\overline{OE}$, and in the limit the current becomes

$$I = \frac{E}{L\omega}.$$

When the circuit contains no ohmic resistance we see, first, that $\overline{CA} = \overline{OA}$, that is, the impressed E. M. F. is equal to $L \omega I$, the E. M. F. of self-induction; and, second, that the current lags 90° behind the impressed E. M. F. These relations, here geometrically shown, are analytically expressed in equation (337).

Self Induction Varied.

Suppose the coefficient of self-induction is varied in a circuit in which the resistance is constant; we wish to find how the current changes.

In the same figure, 52, prolong the line $\overline{EB}$ to F until it meets the impressed E. M. F. $\overline{OA}$ prolonged. Then the line $\overline{BF}$ must equal $\frac{L \omega I}{R}$, since tan BOF equals $\frac{L \omega}{R}$.

The hypotenuse $\overline{OF}$ is, therefore,

$$\overline{OF} = \sqrt{\overline{OB}^2 + \overline{BF}^2} = I \sqrt{1 + \frac{L^2 \omega^2}{R^2}}.$$

But from (29) we find

$$I \sqrt{1 + \frac{L^2 \omega^2}{R^2}} = \frac{E}{R}.$$

$$\text{Therefore,} \qquad \overline{OF} = \frac{E}{R}.$$

Since the hypotenuse $\overline{OF}$ is independent of the current I or the self-induction L, and is a constant for any variation in L, it follows that the current is always represented by a vector, $\overline{OB}$, inscribed in the semi-circle OBF, for any possible variation in the self-induction L. In the figure the arrow shows the direction of change as L increases.

We easily see what the value of the current is in the

limiting cases where L is infinite and zero. When L approaches infinity, the current approaches zero. When L approaches zero, the vector $\overline{OB}$ approaches $\overline{OF}$, the E. M. F. necessary to overcome self-induction is zero, and the current follows Ohm's law, being equal to $\frac{E}{R}$.

That the construction of Fig. 52 is consistent with the equations is further shown by the following relations.

$$(338) \quad \overline{EF}^2 = \left\{ \overline{EB} + \overline{BF} \right\}^2 = \left\{ \frac{R\,I}{L\,\omega} + \frac{L\,\omega\,I}{R} \right\}^2$$

$$= \frac{I^2}{L^2\,\omega^2\,R^2}(R^2 + L^2\,\omega^2)^2$$

$$(339) \quad \overline{OE}^2 + \overline{OF}^2 = \frac{E^2}{L^2\,\omega^2} + \frac{E^2}{R^2} = \frac{E^2}{L^2\,\omega^2\,R^2}(R^2 + L^2\,\omega^2).$$

Equating (338) and (339), we find

$$I^2(R^2 + L^2\,\omega^2) = E^2, \quad \text{or} \quad I = \frac{E}{\sqrt{R^2 + L^2\,\omega^2}},$$

a result which is identical with that analytically expressed in equation (335).

It is seen that in the limiting cases, where the resistance or the self-induction approaches zero or infinity, the triangle of electromotive forces becomes two superimposed straight lines, that is, one side becomes zero. In most of the following problems only the general cases are discussed in which the circuit contains a finite resistance and self-induction. The constructions may be modified, however, according to the principles just set forth, so that the solutions given may be applied to the limiting cases referred to. Although in some cases it may require a little thought and care to make this modification, it has been deemed unnecessary to show its application to each particular problem.

Problem II. Series Circuit. Current Given.

Let there be a circuit, Fig. 53, having n different coils

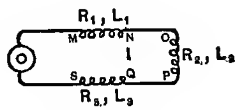


FIG. 53.—PROBLEM II. AND PROBLEM III.

in series, with resistances R_1, R_2, etc., and self-inductions L_1, L_2, etc. It is required to find the impressed E. M. F. necessary to cause a current I to flow through the coils.

In Fig. 54 make $\overline{OA}$ equal to the current flowing.

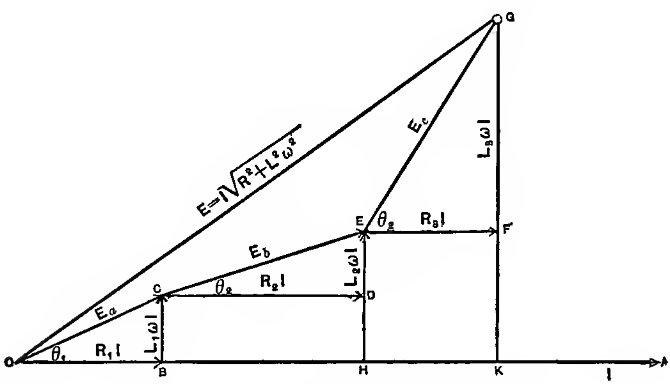


FIG. 54.—PROBLEM II. AND PROBLEM III.

Multiply this by R_1 and lay off $\overline{OB}$ equal $R_1 I$, which is then the effective E. M. F. in the first coil. Draw $\overline{BC}$ perpendicular to $\overline{OA}$ in the positive direction, or direction of advance, and make the angle $B\,O\,C$ equal to $\theta_1 = \tan^{-1}\frac{L_1\,\omega}{R_1}$. Then $B\,O\,C$ is the triangle of E. M. F.'s for coil one, and E_a is its impressed E. M. F. Similarly lay off $\overline{CD}$ parallel to $\overline{OA}$ and equal to $R_2 I$, and then make the angle $D\,C\,E$

equal to $\theta_2 = \tan^{-1} \frac{L_2 \omega}{R_2}$. This triangle CDE then represents the triangle of E. M. F.'s for the second coil, and E_b its impressed E. M. F. In a similar way we may go on constructing triangles of E. M. F.'s for each of the n coils until we finally reach a point G, which is the end of the line representing the impressed E. M. F. in the last coil. If we draw the line $\overline{OG}$, it must be the impressed E. M. F. of the source, which we wished to find, as it is the sum of all the n different falls in potential for each coil. Indeed, this will be evident from the following. If we lay off $\overline{BH} = \overline{CD}$, and $\overline{HK} = \overline{EF}$, we find that $\overline{OK} = R_1 I + R_2 I + \text{etc.} = I \Sigma R$. And, similarly, $\overline{KG} = L_1 \omega I + L_2 \omega I + \text{etc.} = \omega I \Sigma L$. If we replace all the n different coils by a single coil whose resistance is the sum of all the n resistances, viz., ΣR, and whose coefficient of self-induction is the sum of all the n coefficients, viz., ΣL, we find that $\overline{OG}$ is the impressed E. M. F. necessary to cause the given current I to flow, and OKG is the triangle of E. M. F.'s for the equivalent coil.

Problem III. Series Circuit. Impressed E. M. F. Given.

First Method.—The circuit being the same as in PROBLEM II., Fig. 53, it is required to find the current, I, which a given impressed E. M. F. will cause to flow.

We may solve this problem by constructing upon the given E. M. F. $\overline{OG}$, Fig. 54, the triangle OKG so that the angle at O is $\tan^{-1} \frac{\Sigma L \omega}{\Sigma R}$. The side $\overline{OK}$ is then equal to $I \Sigma R$. The current I, $= \overline{OA}$, is then found by dividing $\overline{OK}$ by ΣR.

The impressed E. M. F.'s, E_a, E_b, etc., of the several parts of the circuit are found as in the previous problem.

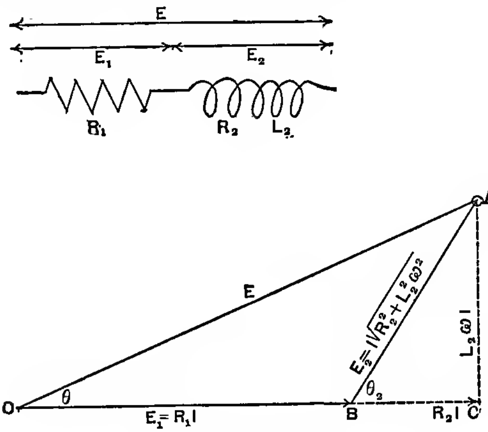
The total effective E. M. F. represented by $\overline{OK}$ is divided in proportion to the resistances R_1, R_2, etc., into the parts $\overline{OB}$, $\overline{BH}$, etc., representing the effective E. M. F. in the several parts of the circuit. The impressed E. M. F.'s are obtained by erecting upon $\overline{OB}$, $\overline{BH}$, etc., the E. M. F. triangles $O\,B\,C$, $C\,D\,E$, etc.

Second Method.—It is sometimes more convenient to solve this problem in the following way. Assume that a certain current is flowing in the circuit; then find the E. M. F. required, by the method of Problem II. Now if the whole figure be magnified or diminished in proportion until the E. M. F. thus found is made equal to the given E. M. F., then the current will be that due to this given E. M. F., which is the required current; for, it is evident that if we change either the E. M. F. or the current, the other is changed in proportion, and indeed the whole diagram is changed in proportion.

Problem IIIa.—Measurements on a Series Circuit.

One of the simplest and also one of the most important cases of series circuits which is met with is that of a non-inductive resistance in series with an inductive resistance as illustrated in Fig. 54*a*. The corresponding diagram of E. M. F.'s is given in Fig. 54*b*, in which $\overline{OB}$ and $\overline{BA}$ represent E_1 and E_2, the E. M. F.'s impressed upon the non-inductive and inductive resistances, respectively, and $\overline{OA}$ represents E, the total impressed E. M. F. The inductive circuit $R_2\,L_2$ may be the primary of a transformer or any inductive circuit whatsoever. From the values of E, E_1, and E_2, which are readily obtained from three voltmeter readings, and the value of the non-inductive resistance R_1, we can ascertain the following quantities: the angle θ by which the current lags behind the impressed E. M. F., E; the angle θ_2 by which the current in the inductive part of

the circuit lags behind the E. M. F., E_2, impressed upon that part; the impedance, resistance, and self-induction of the inductive circuit (in the case of a transformer it is the *apparent* resistance and self-induction which is found); and



FIGS. 54*a*, 54*b*.

the power expended in each part of the circuit and in the whole circuit.

From the values E, E_1, and E_2, the triangle $O\,A\,B$ is drawn, and upon $\overline{O\,A}$ the right triangle $O\,C\,A$ is erected by producing $\overline{O\,B}$ to C.

The resistance R_2 is obtained thus. $\overline{O\,B} = R_1\,I$, $\overline{B\,C} = R_2\,I$. Therefore, $R_1 : R_2 :: \overline{O\,B} : \overline{B\,C}$. For R_1 we may write $\frac{E_1}{I}$. The resistance R_2 is then

$$R_2 = \frac{\overline{B\,C}}{\overline{O\,B}}\,R_1 = \frac{\overline{B\,C}}{\overline{O\,B}}\,\frac{E_1}{I}.$$

The angle θ by which the current lags behind the E. M. F. impressed upon the whole circuit is found from the geometry of the figure. By trigonometry,

$$E_2^2 = E^2 + E_1^2 - 2\,E E_1 \cos\theta.$$

Whence $\cos\theta = \frac{E_2^2 + E_1^2 - E_2^2}{2\,E\,E_1}.$

The angle θ_2, by which the current lags behind the E. M. F. impressed upon the inductive part of the circuit is similarly found from the trigonometrical expression

$$E^2 = E_1^2 + E_2^2 - 2\,E_1\,E_2 \cos O\,B\,A.$$

From this it follows that

$$\cos\theta_2 = -\cos O\,B\,A = \frac{E^2 - E_1^2 - E_2^2}{2\,E_1\,E_2}.$$

In the non-inductive portion of the circuit, the current is in phase with the E. M. F. and $\theta_1 = 0$.

The value of the self-induction of the inductive circuit is obtained from the values for R_2 and θ_2 given above, and from the relation $\tan\theta_2 = \frac{L_2\,\omega}{R_2}$. The value of the expression $L_2\,\omega$, called the inductive resistance in contradistinction to ohmic resistance, may be given in ohms. To find L_2, θ_2 is first found from the expression given above for $\cos\theta_2$ by means of trigonometry tables, and the tangent of the angle is then found, also from the tables, and equated to $\frac{L_2\,\omega}{R_2}$, from which L_2 may be readily calculated.

An explicit expression for L_2 in terms of E_1, E_2, and E may be found as follows. From the figure it is seen that

$$L_2 = \frac{E_2}{I\,\omega}\sin\theta_2 = \frac{E_2}{I\,\omega}\sqrt{1 - \cos^2\theta_2}.$$

When the expression given above for θ_2 is substituted in this expression, it becomes

$$L_2 = \frac{1}{2\,I\,\omega\,E_1}\sqrt{2E_1^2E_2^2 + 2E^2E_1^2 + 2\,E^2E_2^2 - E^4 - E_1^4 - E_2^4}.$$

Inasmuch as the expression involves the differences of the fourth powers, it does not afford as accurate a method for determining self-induction as that given in the preceding paragraph.

In these expressions, E, E_1, E_2, and I represent maximum values, but in the above cases the expressions would be the same if the *virtual* values were used, that is, the square root of the mean square of the instantaneous values [see page 38], which are represented by $\overline{I}$, $\overline{E}$, etc. This is because the values of the above expressions all depend upon the ratio of the quantities in such a way that if each quantity were multiplied by the same constant, the values of the expressions themselves would remain unchanged. It is therefore immaterial whether maximum or virtual values are used.

In obtaining the expressions for the power expended in each portion of the circuit the virtual values will be used, inasmuch as these are the values usually obtained from alternating-current measuring instruments. The general expression for the power imparted to a circuit is [see (195) and (196)]

$$W = \tfrac{1}{2} E\, I \cos \theta = \overline{E}\, \overline{I} \cos \theta,$$

where θ is the angle of lag between the E. M. F. and the current.

In the non-inductive resistance the angle of lag is zero and the power is, therefore,

$$W_1 = \overline{E}_1 \overline{I}.$$

The power expended in the inductive part of the circuit is

$$W_2 = \overline{E}_2 \overline{I} \cos \theta_2 = \frac{\overline{I}}{2\,\overline{E}_1}(\overline{E}^2 - \overline{E}_1^2 - \overline{E}_2^2)$$

$$= \frac{1}{2\,R_1}(\overline{E}^2 - \overline{E}_1^2 - \overline{E}_2^2).$$

The power expended in the whole circuit is

$$W = \overline{E}\ \overline{I} \cos\theta = \frac{\overline{I}}{2\,\overline{E}_1}\left(\overline{E}^2 + \overline{E}_1^2 - \overline{E}_2^2\right)$$

$$= \frac{1}{2R_1}\left(\overline{E}^2 + \overline{E}_1^2 - \overline{E}_2^2\right).$$

It is evident that the power expended in the whole circuit is the sum of the power in each part, or

$$W = W_1 + W_2.$$

This method of measuring power is known as the three-voltmeter method and was apparently suggested by Mr. Swinburne and by Prof. Ayrton and Dr. Sumpner at about the same time. The method is applicable to any circuit whether the E. M. F. is harmonic or not.* For maximum accuracy $E_1 = E_2$.

Problem IV. Divided Circuit. Two Branches. Impressed E. M. F. Given.

Let us consider the problem of a divided circuit having two branches in parallel as indicated in Fig. 55. Each

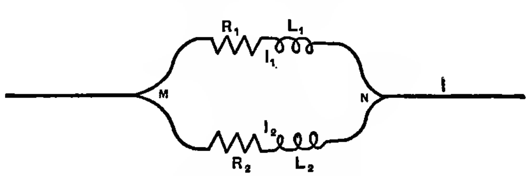


FIG. 55.—PROBLEM IV.

branch contains self-induction and resistance, and there is an impressed E. M. F., E, between the terminals M and N;

* See "The Measurement of the Power given by any Electric Current to any Circuit:" Prof. Ayrton and Dr. Sumpner; *Proc. Roy. Soc.*, Vol. XLIX., 1891, p. 424.

it is required to find the main current, I, and the currents I_1 and I_2 in the branches.

Fig. 56 shows how to find graphically the main and branch currents when the impressed E. M. F. and the resistance and self-induction of each branch are given.

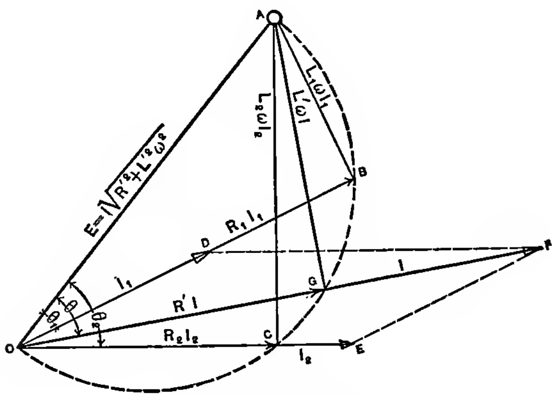


FIG. 56.—PROBLEM IV.

Since the impressed E. M. F. at the terminals of each branch is known, each may be separately treated as a simple circuit containing resistance and self-induction by the method previously given in Fig. 50.

Draw $\overline{OA}$ equal to the impressed E. M. F., E. Make the angle $A\,O\,B = \theta_1 = \tan^{-1}\frac{L_1\,\omega}{R_1}$ in the negative direction such that it is an angle of lag. Then the right triangle $O\,B\,A$ is the triangle of E. M. F.'s for the first branch, $\overline{OB}$ is the E. M. F. necessary to overcome resistance, and $\overline{BA}$ that necessary to overcome the self-induction. In a similar way we may lay off the angle $A\,O\,C = \theta_2 = \tan^{-1}\frac{L_2\,\omega}{R_2}$ to represent the angle of lag in the second branch, and then construct the triangle $O\,C\,A$, which will represent the E. M. F.'s in the second branch. Since these are right

triangles, the points B and C lie on the circumference of a circle whose diameter is $\overline{OA}$. Since the effective E. M. F., $R_1 I_1$, in the first branch is $\overline{OB}$, the current is $\overline{OD}$, equal to $\overline{OB}$ divided by R_1. Similarly the current I_2 is $\overline{OE}$, equal to $\overline{OC}$ divided by R_2. Now the current in the main circuit at any instant is equal to the sum of the currents in the branch circuits at that instant. Construct, therefore, the parallelogram upon the sides $\overline{OD}$ and $\overline{OE}$. The diagonal $\overline{OF}$ represents the main current, I, for its projection at any moment equals the sum of the projections of the two sides $\overline{OD}$ and $\overline{OE}$, which projections represent the instantaneous values of the current in the two branches.

From the geometry of the figure it follows that

$$E = I_1 \sqrt{R_1^2 + L_1^2 \omega^2} = I_2 \sqrt{R_2^2 + L_2^2 \omega^2} = I \sqrt{R'^2 + L'^2 \omega^2}.$$

It is seen that the current in each branch is inversely proportional to the impedance.

This diagram gives the complete solution of the problem of the divided circuit. The currents I_1 and I_2 in the branches lag behind the impressed E. M. F., E, by angles θ_1 and θ_2. The main current, I, lies between these, making an angle θ with E. It is evident that the maximum value of the main current, I, being the longest diagonal of the parallelogram whose sides represent the currents in the branches, is greater than the current in either branch. Since the currents differ in phase, at certain parts of a period it happens that the current in a branch is greater than the main current, for when the main current is zero the branch current may have a considerable value.

Equivalent Resistance and Self Induction.

Suppose that instead of the two parallel branches which have been considered, a single circuit be substituted for them whose resistance, R', and self-induction, L', are such that

the same current as before will flow in the main line. Then $O\,G\,A$ must represent the triangle of E. M. F.'s for this circuit, since the impressed E. M. F. is $\overline{OA}$, and the effective E. M. F. is $\overline{OG}$, in the direction of the current, and the E. M. F. $\overline{GA}$ to overcome self-induction is at right angles to the current. The resistance, R', and self-induction, L', of the equivalent simple circuit—that is, a circuit which allows the same current to flow in the main line—are called the equivalent resistance and equivalent self-induction of the divided circuit.

The values of this equivalent resistance, R', and self-induction, L', may easily be found in terms of the resistances and self-inductions of the branches. This will be deferred until after the discussion of the following problem, in which the solution is given for any number of circuits connected in parallel.

Problem V. Divided Circuit. Any Number of Branches. Impressed E. M. F. Given.

Let the divided circuit MN, Fig. 57, have n branches

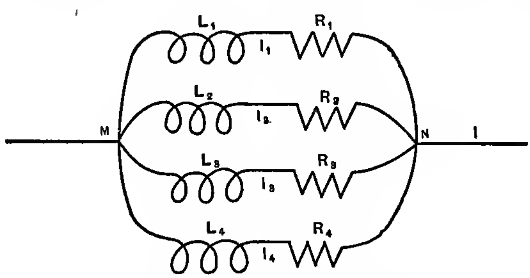


FIG. 57.—PROBLEM V. AND PROBLEM VI.

in parallel, each containing resistance and self-induction, with an impressed E. M. F., E, between the terminals M and N. The currents I_1, I_2, . . . I_n in each branch may be constructed as in Problem IV., where there were only two

branches, and the resultant current, I, in the main line found, since it is the geometrical resultant of the n branch currents. Fig. 58 is constructed as follows. Draw a semi-

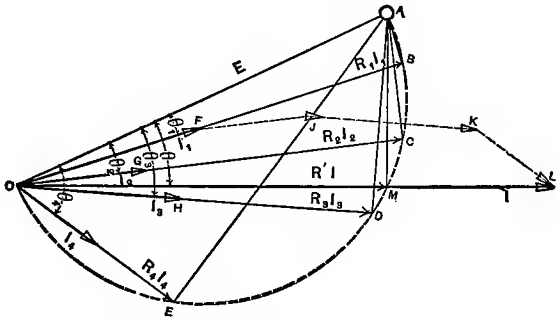


FIG. 58.—PROBLEM V. AND PROBLEM VI.

circle upon the impressed E. M. F., $\overline{OA}$, and lay off the n different angles θ_1, θ_2, . . . θ_n in the negative or lag direction, which represent the lag of the current in each branch behind the impressed E. M. F., E. This will give n different right triangles $O\,B\,A$, $O\,C\,A$, $O\,D\,A$, etc., which represent the E. M. F.'s in each branch, the sides of which represent the effective E. M. F., the E. M. F. to overcome self-induction, and the impressed E. M. F. Now the currents I_1, I_2, I_3, etc., or $\overline{O\,F}$, $\overline{O\,G}$, $\overline{O\,H}$, etc., are found by dividing the effective E. M. F.'s $R_1\,I_1$, $R_2\,I_2$, $R_3\,I_3$, etc., or $\overline{O\,B}$, $\overline{O\,C}$, $\overline{O\,D}$, etc., by the resistances R_1, R_2, R_3, etc. The resultant current, I, or $\overline{O\,L}$, is found by taking the geometrical resultant of all the branch currents, $\overline{O\,F}$, $\overline{O\,G}$, $\overline{O\,H}$, etc. This construction is shown by the closed polygon $O\,F\,J\,K\,L$, each side of which is equal to a branch current. By this construction it is evident that, since the angles $O\,F\,J$, $F\,J\,K$, etc., must be each greater than a right angle, the maximum resultant current, I, or $\overline{O\,L}$, is greater than any of the branch currents. During a certain portion of each period, as before explained, the instantaneous value

of the resultant current is less than the instantaneous value of the current in any one branch.

Equivalent Resistance and Self-induction of Parallel Circuits.

In this case, as in the previous one, suppose that a single equivalent circuit is substituted for the n parallel branches having such a resistance, R', and self-induction, L', that the current in the main line is not changed either in magnitude or phase. The values of this equivalent resistance, R', and equivalent self-induction, L', may easily be found in terms of the resistances and self-inductions of each branch. In Fig. 58 the triangle $O\,M\,A$ must represent the triangle of E. M. F.'s for the single equivalent circuit substituted for the system of parallel branches, if the resultant current $\overline{O\,L}$ is to be the same; for, the effective E. M. F., $R'I$, is in the direction of the current $\overline{O\,L}$, and is therefore equal to $\overline{O\,M}$, since the E. M. F. to overcome the self-induction, $L'\omega I$ or $\overline{M\,A}$, is perpendicular to the current.

To find R' and L', as well as the tangent of the angle θ which the main current makes with the impressed E. M. F., we may proceed as follows.

If we take the projections of the currents I, I_1, I_2, etc., upon the line $\overline{O\,A}$, we obtain the equation

$$(340) \quad I\cos\theta = I_1\cos\theta_1 + I_2\cos\theta_2 + \ldots = \sum I\cos\theta.$$

If we consider the projections of the currents upon a line perpendicular to $\overline{O\,A}$, we obtain

$$(341) \quad I\sin\theta = I_1\sin\theta_1 + I_2\sin\theta_2 + \ldots = \sum I\sin\theta.$$

Since all the triangles $O\,B\,A$, $O\,C\,A$, etc., are right

triangles, the following values for I, I_1, I_2, etc., and for $\cos\theta$, $\cos\theta_1$, etc., $\sin\theta$, $\sin\theta_1$, etc., will be evident.

(342)
$$I = \frac{E}{\sqrt{R'^2 + L'^2\omega^2}},$$

$$I_1 = \frac{E}{\sqrt{R_1^2 + L_1^2\omega^2}},$$

$$I_2 = \frac{E}{\sqrt{R_2^2 + L_2^2\omega^2}}, \qquad \text{etc.}$$

(343)
$$\cos\theta = \frac{R'}{\sqrt{R'^2 + L'^2\omega^2}},$$

$$\cos\theta_1 = \frac{R_1}{\sqrt{R_1^2 + L_1^2\omega^2}},$$

$$\cos\theta_2 = \frac{R_2}{\sqrt{R_2^2 + L_2^2\omega^2}}, \qquad \text{etc.}$$

(344)
$$\sin\theta = \frac{L'\omega}{\sqrt{R'^2 + L'^2\omega^2}},$$

$$\sin\theta_1 = \frac{L_1\omega}{\sqrt{R_1^2 + L_1^2\omega^2}},$$

$$\sin\theta_2 = \frac{L_2\omega}{\sqrt{R_2^2 + L_2^2\omega^2}}, \qquad \text{etc.}$$

Substituting these values in (340), we have

(345)
$$\frac{I\cos\theta}{E} = \frac{R'}{R'^2 + L'^2\omega^2}$$

$$= \frac{R_1}{R_1^2 + L_1^2\omega^2} + \frac{R_2}{R_2^2 + L_2^2\omega^2} + \ldots = \sum\frac{R}{R^2 + L^2\omega^2}.$$

Making a similar substitution in (341), we have

$$(346) \quad \frac{I \sin \theta}{E} = \frac{L' \omega}{R'^2 + L'^2 \omega^2}$$

$$= \frac{L_1 \omega}{R_1^2 + L_1^2 \omega^2} + \frac{L_2^2 \omega}{R_2^2 + L_2^2 \omega^2} + \dots = \sum \frac{L \omega}{R^2 + L^2 \omega^2}$$

For brevity, let

$$(347) \qquad \sum \frac{R}{R^2 + L^2 \omega^2} = A,$$

$$(348) \qquad \text{and} \quad \sum \frac{L \omega}{R^2 + L^2 \omega^2} = B \omega.$$

Dividing (346) by (345), we have

$$(349) \qquad \tan \theta = \frac{B \omega}{A}.$$

From equations (345) and (346), we have

$$\frac{R'}{R'^2 + L'^2 \omega^2} = A,$$

$$\text{and} \quad \frac{L' \omega}{R'^2 + L'^2 \omega^2} = B \omega.$$

Comparing these with the values of cos θ and sin θ in (343) and (344), we obtain

$$(350) \qquad A = \frac{\cos^2 \theta}{R'}, \quad \text{or} \quad R' = \frac{\cos^2 \theta}{A},$$

$$(351) \quad \text{and} \quad B \omega = \frac{\sin^2 \theta}{L' \omega}, \quad \text{or} \quad L' \omega = \frac{\sin^2 \theta}{B \omega}.$$

For $\cos^2\theta$ and $\sin^2\theta$ we may substitute the values

$$\cos^2\theta = \frac{1}{1+\tan^2\theta} = \frac{1}{1+\dfrac{B^2\omega^2}{A^2}} = \frac{A^2}{A^2+B^2\omega^2}.$$

$$\sin^2\theta = \frac{1}{1+\cot^2\theta} = \frac{1}{1+\dfrac{A^2}{B^2\omega^2}} = \frac{B^2\omega^2}{A^2+B^2\omega^2}.$$

Making these substitutions, equations (350) and (351) become

(352) $$R' = \frac{A}{A^2+B^2\omega^2}.$$

(353) $$L'\omega = \frac{B\omega}{A^2+B^2\omega^2}.$$

These expressions, (352) and (353), enable us to calculate the equivalent resistance and self-induction of any number of parallel circuits when we know the resistance and self-induction of each. The angle of lag of the main current is found from (349). These same analytical results were otherwise obtained by Lord Rayleigh, and given by him in a paper on "Forced Harmonic Oscillations of Various Periods" in the *Philosophical Magazine,* May 1886. The present demonstration was first given by the authors in the *Philosophical Magazine* for September 1892.

Problem VI. Divided Circuit. Current Given.

Suppose we have a number of circuits, each containing resistance and self-induction, connected in parallel as in Fig. 57, and we know the value of the current, I, in the main line. It is required to find the current in each of the several branches. The value of the impressed E. M. F. is not known, and so the construction cannot be made in the same manner as in the problem just discussed.

First Method. Entirely Graphical.

We can, however, assume any value for the impressed E. M. F., E, and make the construction accordingly, as in the previous problem. We would thus obtain a value for the main current, I, different from the one given. The diagram will be correct in all respects except the scale, and this must be changed in the ratio of the given value of I to the value of I obtained from the assumed impressed E. M. F. The true value of the impressed E. M. F. and the current in each branch may thus be obtained and the solution is complete.

Second Method. Solution by Use of Equivalent Resistance and Self Induction.

Another solution for this same problem is obtained by the use of equivalent resistance and equivalent self-induction of parallel circuits. These values for R' and L' are calculated according to the expressions (352) and (353). Draw $\overline{OM}$, Fig. 58, equal to $R'I$, and draw $\overline{MA}$ perpendicular to $\overline{OM}$ and equal to $L'\omega I$. The hypotenuse $\overline{OA}$ of the right triangle OMA gives us the value of the impressed E. M. F., E. The further construction is the same as before. The angles of lag θ_1, θ_2, θ_3, etc., are laid off, and the E. M. F. triangle for each branch circuit is drawn. The effective E. M. F. and the current in each branch are thus readily found.

Problem VII. Effects of the Variation of the Constants R and L in a Divided Circuit of Two Branches.

Resistance Alone Varied in Either Branch.

Suppose the resistance of one branch of a divided circuit to be varied and the other constants to remain unchanged; it is required to find the changes in the currents due to this variation in resistance when there is a constant

impressed E. M. F. Let the diagram for the divided circuit shown in Fig. 55 be represented in Fig. 59, where the same letters represent the same points as in the diagram, Fig. 56, already given for the divided circuit.

If the resistance R_1 is varied, it is evident that the effective E. M. F. $\overline{OB}$ always lies on the semi-circle $O\,B\,A$, and, as this branch may be regarded as a single circuit having a constant E. M. F. and a resistance which is varied, the cur-

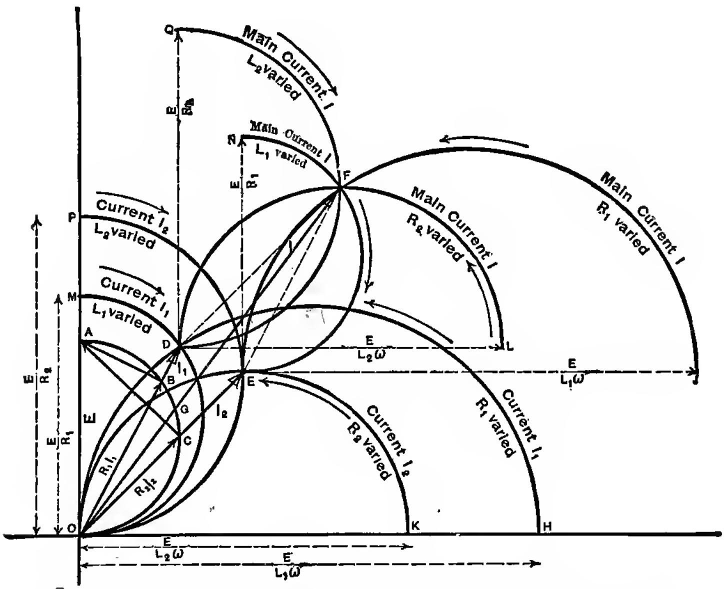


FIG. 59.—VARIATION OF RESISTANCE AND SELF-INDUCTION IN A DIVIDED CIRCUIT. PROBLEM VII.

rent I_1 always lies on the semi-circle $O\,D\,H$, whose diameter is $\overline{OH}$ or $\frac{E}{L_1\,\omega}$ (see PROBLEM I.). If R_1 is the only quantity varied, it is evident that the resultant main current must lie on the semi-circle $E\,F\,J$, whose diameter $\overline{EJ}$ is equal to $\overline{OH}$.

Similarly, if R_2 is varied alone, the current I_2 must lie

on the semi-circle OEK, whose diameter is $\overline{OK}$ or $\frac{E}{L_2\omega}$. The resultant main current will then lie on the semi-circle DFL. When both resistances are varied at the same time the currents I_1 and I_2 lie on their semi-circles ODH and OEK; but the resultant or main current has no particular locus.

The arrows on the curves, showing the effects of a variation of the resistance, indicate the direction of the change as the resistance increases.

Self Induction Alone Varied in Either Branch.

Regarding each branch of the divided circuit, having a constant difference of potential at its terminals, as a single circuit, it is evident that any variation of L_1 alone will cause the current vector I_1 to lie upon the semi-circle ODM, whose diameter is $\frac{E}{R_1}$ (see Problem I.). Any variation of L_1 alone will cause the resultant main current vector, I, to lie upon the semi-circle EFN.

Similarly, when L_2 alone is varied, the current I_2 lies upon the semi-circle OEP, and the resultant current I upon the semi-circle DFQ. If both L_1 and L_2 are simultaneously changed, the currents I_1 and I_2 still lie on their circles ODM and OEP, respectively, but the resultant current I has no particular locus.

The arrows on the curves, showing the effects of a variation of the self-induction, indicate the direction of the change as the self-induction increases.

Limiting Cases.

This diagram enables us to see what the currents will be in the divided circuit in the limiting cases when the resistances or self-inductions approach infinite or zero values. As a particular instance, suppose it happens that

L_2 is zero, and R_1 is very small compared with L_1. This means that there is self-induction alone in one branch and resistance alone in the other. The current I_1 would then be represented by $\overline{OH}$, and I_2 by $\overline{OP}$, and the main current, I, by the resultant of these.

Constant Potential Example.—As an example, suppose there is an incandescent lamp, Fig. 60, of 50 ohms resistance R_2, and a coil whose self-induction L_1 is .5 henrys shunted around the lamp, the terminals of which are subjected to a constant difference of potential of 50 volts. What are the currents through the lamp, coil, and

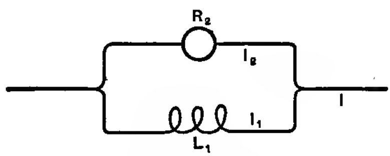


FIG. 60.—PROBLEM VII.

the main line? Suppose that $\omega = 1000$. We may calculate

$$\frac{E}{R_2} = \frac{50}{50} = 1 = I_2,$$

$$\text{and} \quad \frac{E}{L_1 \omega} = \frac{50}{.5 \times 1000} = .1 = I_1.$$

Make $\overline{OP}$, Fig. 61, equal to $\dfrac{E}{R_2} = 1$, and ninety degrees behind it make $\overline{OH} = \dfrac{E}{L_1 \omega} = .1$. The resultant current $\overline{OS}$ is easily calculated, thus:

$$\overline{OS} = \sqrt{\overline{OP}^2 + \overline{OH}^2} = \sqrt{1 + .01} = 1.005, \text{ approx.}$$

If the incandescent lamp should break, the current I_2 through it would be stopped and the main current reduced to $\overline{OH}$, equal to .1 amperes.

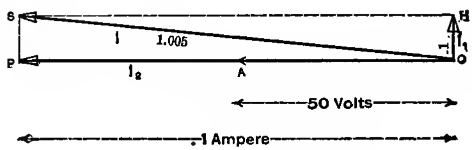


FIG. 61.—CONSTANT POTENTIAL EXAMPLE, PROBLEM VII.

Constant Current Example.—Suppose that, instead of being subjected to a constant potential, a divided circuit, as Fig. 60, is supplied with a constant current. Let the main current be maintained constantly at ten amperes. It is required to find the branch currents and the difference of potential at the terminals. Let it have a resistance R_2 of two ohms, and let the self-induction of the choke-coil be. 02

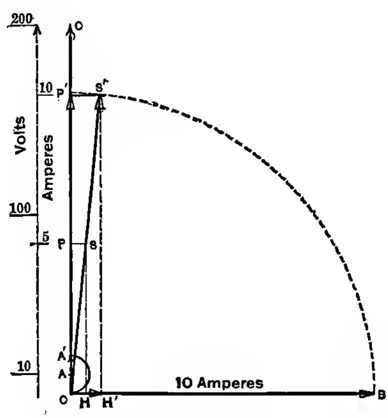


FIG. 62.—CONSTANT CURRENT EXAMPLE, PROBLEM VII.

henrys. Using the first method of solving the problem of the divided circuit when the current is given, Problem VI., we may assume an impressed E. M. F. $\overline{OA}$ of ten volts. Following the same construction in Fig. 62 as in Fig. 61 for

the solution of the constant potential example, we may calculate

$$\frac{E}{R_2} = \frac{10}{2} = 5 = I_2 = \overline{OP},$$

and $$\frac{E}{L_1\,\omega} = \frac{10}{.02 \times 1000} = .5 = I_1 = \overline{OH}.$$

The resultant $\overline{OS}$ is calculated thus:

$$\overline{OS} = \sqrt{\overline{OP}^2 + \overline{PS}^2} = \sqrt{25 + .25} = 5.025 \text{ amperes.}$$

Since the main current should be ten amperes, it is necessary to magnify the whole diagram in the ratio $\frac{10}{5.025}$, in order to find the true difference of potential at the terminals, and the true branch currents. This makes the impressed E. M. F. $10 \times \frac{10}{5.025}$ equal to 19.9 volts; the current I_2 equal to 9.95 amperes; and I_1 equal to .995 amperes. Since the current and the E. M. F. are in phase, the energy consumed by the lamp is equal to $19.9 \times 9.95 = 198$ watts. The energy consumed by the choke-coil is almost nothing, since the current I_1 is almost at right angles to the impressed E. M. F.

If the lamp filament should break, the current I_2 would be suddenly stopped and the whole current $\overline{OB}$ of ten amperes would flow through the coil. The potential $\overline{OC}$ at the terminals would suddenly become much greater, large enough to overcome the E. M. F. of self-induction $L_1\,\omega\,I$, that is, $.02 \times 1000 \times 10 = 200$ volts.

Thus the choke-coil shunted around the lamp consumes but little energy and prevents the current from being interrupted when the lamp breaks. In case the lamp does break, however, there is the sudden rise in potential as shown above.

CHAPTER XVI.

PROBLEMS WITH CIRCUITS CONTAINING RESISTANCE AND SELF-INDUCTION.

COMBINATION CIRCUITS.

Problem VIII. Series and Parallel Circuits. Impressed E. M. F. Given. Solution by Use of Equivalent Resistance and Self Induction.

PROBLEMS arising from combinations of series and parallel circuits are readily solved by the repeated application of the foregoing methods. Let us consider the case where two systems of parallel circuits are joined in series, as in Fig. 63. The resistance and self-induction of each branch is given and the total impressed E. M. F. It is required to find the current in the main line and in the branches.

The equivalent resistance and self-induction R_a' and L_a' between M and N, and R_b' and L_b' between N and O, are readily found according to the formulæ (352) and (353). We can now treat the problem as that of a series circuit, as

in PROBLEM III., and ascertain the impressed E. M. F. between M and N and between N and O.

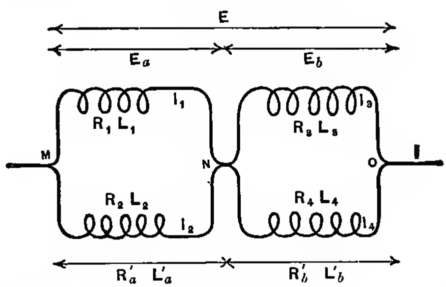


FIG. 63.—PROBLEM VIII. AND PROBLEM IX.

Upon the impressed E. M. F., $\overline{OA}$, Fig. 64, draw the right triangle OBA such that $\tan AOB = \dfrac{(L_a' + L_b')\omega}{R_a' + R_b'}$.

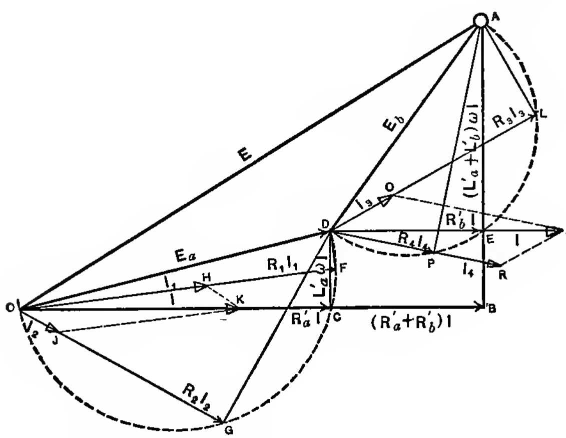


FIG. 64.—PROBLEM VIII. AND PROBLEM IX.

Then $\overline{OB}$ is the E. M. F. effective in overcoming the resistance $R_a' + R_b'$ and may be divided at C so as to show the E. M. F. effective in overcoming each.

$$\overline{OC} : \overline{CB} :: R_a' : R_b'.$$

$\overline{OC}$ is the E. M. F. effective in overcoming the resistance R_a'. Draw $\overline{CD}$ perpendicular to $\overline{OC}$, and complete the right triangle OCD, so that $\tan DOC = \frac{L_a'\omega}{R_a'}$. $\overline{OD}$ is the impressed E. M. F. between the points M and N, and $\overline{DA}$ the impressed E. M. F. between N and O. Knowing the impressed E. M. F., E_a, between M and N, we can obtain the currents I_1 and I_2 in the branches by the method fully explained in PROBLEM IV. and PROBLEM V. On the diameter $\overline{OD}$, the E. M. F. triangles OFD and OGD are drawn with angles of lag according to the constants of each branch. The currents I_1, I_2, and I are found by dividing the effective E. M. F.'s by the resistances R_1, R_2, and R_a', respectively. In the same way, the E. M. F. triangles DLA and DPA are erected on the line $\overline{DA}$, which represents E_b, the effective E. M. F. between N and O. The currents I_3 and I_4 are then found and we have the complete solution of the problem.

Problem IX. Series and Parallel Circuits. Current Given. Solution by Use of Equivalent Resistance and Self-induction.

Suppose that we have the same arrangement of circuits as that just described and shown in Fig. 63, and that the main current, I, is given. It is required to find the current in each branch. The solution for the part between M and N and for the part between N and O can be obtained independently according to the second method given in PROBLEM VI.

In Fig. 64, $\overline{OC}$ is drawn equal to $R_a'I$, and $\overline{CD}$ equal to $L_a'\omega I$. On $\overline{OD}$ the E. M. F. triangles $\overline{OFD}$, $\overline{OGD}$ are drawn and the solution for branches one and two obtained. $\overline{DE}$ is then drawn parallel to $\overline{OC}$ and equal to $R_b' I$, and $\overline{EA}$ equal to $L_a'\omega I$. The E. M. F. triangles, DLA and

DPA, are then erected on $\overline{DA}$, and the solution for branches three and four obtained in the regular way. The line connecting O and A gives the total impressed E. M. F., E.

Problem X. Extension of Problems VIII. and IX.

The solution given in PROBLEM VIII. may be applied to any combination of circuits in series and parallel. Let us consider a combination of circuits such as that shown in

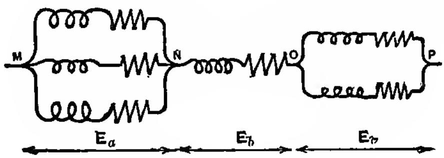


FIG. 65.—PROBLEM X.

Fig. 65, having a given impressed EMF between the points M and P. The circuits may be divided into three parts, MN, NO, and OP, and the equivalent resistance and self-induction of each obtained [see (352) and (353)]. The impressed E. M. F's., E_a, E_b, E_c, of each portion can now be laid off, Fig. 66, according to the method given for series

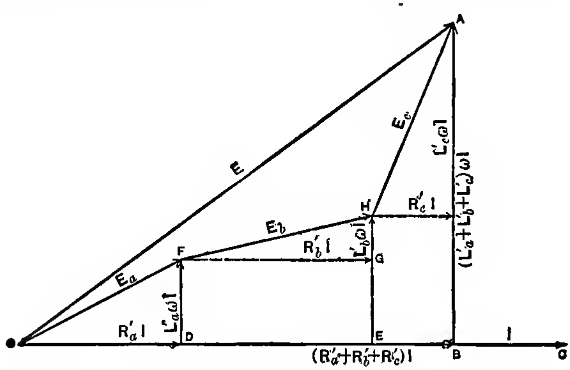


FIG. 66.—PROBLEM X.

circuits, PROBLEM III., and a semi-circle erected upon each, as was done upon E_a and E_b in Fig. 64. Each portion of the

circuit is now treated separately according to the method for parallel circuits, PROBLEM V. In each semi-circle the various E. M. F. triangles are drawn and the currents in the several branches found.

If the main current is given in an extended system of conductors, as in Fig. 65, the solution is obtained, as in PROBLEM VII., by dividing the system into its several sets of parallel circuits and treating the separate sets, *M N*, *N O*, *O P*, independently.

Problem XI. Series and Parallel Circuits. Entirely Graphical Solution.

In the previous treatment of the problems arising from combinations of circuits in series and parallel it was necessary to find analytically the values of the equivalent resistance and self-induction of each set of parallel circuits, and the solutions were, therefore, partly analytical and partly graphical. They may be obtained, however, by entirely graphical methods, if we assume some value for the current in a particular branch or assume its impressed E. M. F., solve a portion of the system of conductors accordingly, and then correct the scale as required by the given conditions of the problem. Various ways of doing this suggest themselves as preferable according to the nature of the problem.

BY ASSUMING VARIOUS IMPRESSED E. M. F.'S.

Given the main current, I, in a system as shown in Fig. 67. Let us assume any value we please for the impressed

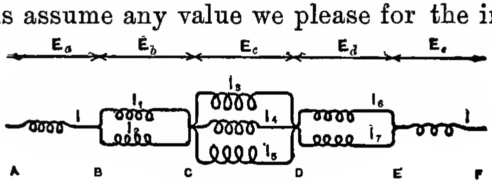


FIG. 67.—PROBLEM XI.

E. M. F., E_b, then erect, on a line representing E_b, the E. M. F. triangles for branches one and two, and thus find

the currents I_1, I_2, and I, due to this assumed E. M. F. The value thus obtained for the main current, I, will be different from the given value, and the assumed E. M. F., E_b, must be changed in the ratio of the given value of I to the value obtained from the assumed value of E_b. This amounts to changing the scale of the drawing. The solution is thus obtained for the part between B and C. The several other portions CD, DE of the system are separately treated in the same manner and thus a complete solution obtained.

If we were given the total impressed E. M. F., E, and not the main current, I, a convenient graphical solution would be obtained by assuming some value for I, solving as in the previous paragraph, and then changing the scale according to the ratio of the given value of the impressed E. M. F. to the value thus obtained.

By Assuming the Current in Certain Branches.

Instead of assuming the E. M. F. impressed at the terminals of each part of the system, we may assume the current flowing in any one branch of each parallel set of conductors. The complete graphical solution by this method of a combination circuit representing in Fig. 68 is given,

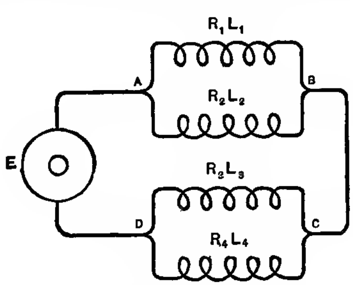


Fig. 68.—Problem XI.

to illustrate the principles already given. The total impressed E. M. F., E, is given.

In Fig. 69 any assumed line $\overline{OA}$ is drawn to represent the current in the branch whose resistance and self-induc-

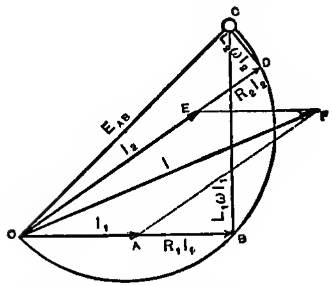


FIG. 69.—PROBLEM XI.
THE SOLUTION FOR CIRCUITS BETWEEN *A* AND *B*, FIG. 68

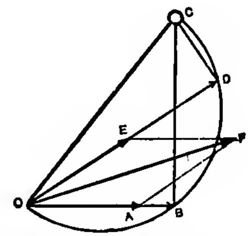


FIG. 70.—PROBLEM XI.
THE SOLUTION FOR CIRCUITS BETWEEN *C* AND *D*, FIG. 68.

tion are R_1 and L_1. $\overline{OA}$ is then multiplied by R_1 and produced to B. Then $\overline{OB} = R_1 I_1$ is the effective E. M. F. in branch one. Draw $\overline{BC}$ perpendicular to $\overline{OB}$ in the direction of advance and make it equal to $L_1 \omega I_1$. Then $\overline{OC}$ is the impressed E. M. F. necessary to drive the assumed current through branch one. Now having this impressed E. M. F., we can draw the E. M. F. triangle, ODC, for branch two, and obtain the current $\overline{OE}$ flowing in the second branch by dividing $\overline{OD}$ by R_2. The total current $I = \overline{OF}$ is then the vector sum of I_1 and I_2, or of $\overline{OA}$ and $\overline{OE}$.

For the parallel system between D and C (Fig. 68), the same process is followed, and in Fig. 70 $\overline{OA}$ is first assumed as the current branch in three, and $\overline{OF}$ finally found to be the total current flowing between C and D (Fig. 68). Since these two parallel systems are in series, the total current, $\overline{OF}$ (Fig. 69), flowing between A and B (Fig. 68), must equal the total current, $\overline{OF}$ (Fig. 70), flowing between C and D (Fig. 68). Hence Fig. 70 is magnified until $\overline{OF}$ becomes as large as $\overline{OF}$, Fig. 69, and is represented in Fig. 71. Next the two figures 69 and 71 are combined, as

shown in Fig. 72, so that $\overline{O'F'}$ is parallel with $\overline{OF}$, since each represents the same current I. $\overline{OC'}$, the vector sum of

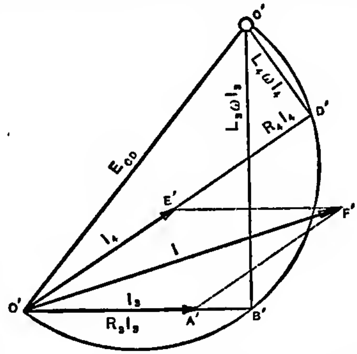


FIG. 71.—PROBLEM XI. FIG. 70 ENLARGED.

$E_{\overline{ab}}$ and $E_{\overline{cd}}$, is the total impressed E. M. F. at the terminals A and D necessary to send the current I. If this

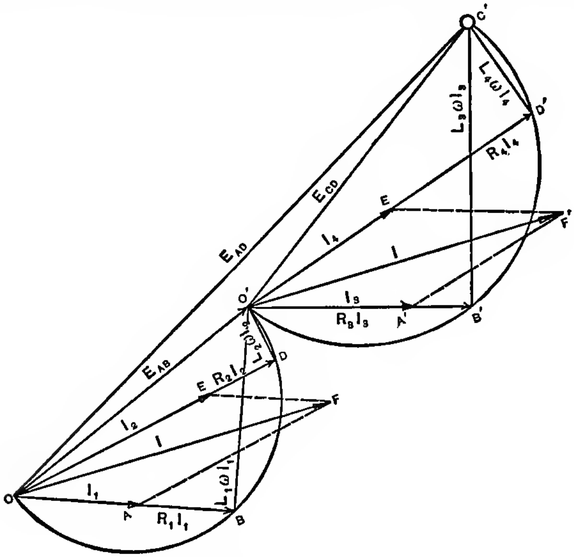


FIG. 72.—PROBLEM XI.

figure is now magnified until $\overline{OC'}$ is equal to the given impressed E. M. F., the solution of the problem is complete

and we have found the currents in each branch for the given impressed E. M. F.

Problem XII. Multiple-arc Arrangement.

Graphical solutions for circuits in series and for circuits in parallel have been separately explained at length and it has been shown how the solution of any combination of circuits in series and parallel may be obtained by dividing the system into its separate parts of series and parallel arrangements and successively applying the foregoing methods. There are countless combinations which might arise, but the solutions of all depend upon the principles already given, and it will suffice to further illustrate them by their application to one more problem of combined circuits.

Let us consider a system of parallel circuits, each with resistance and self-induction, extending between two mains containing resistance and self-induction. Such a system is shown in Fig. 73. The circuits 1, 2, 3, etc., contain resist-

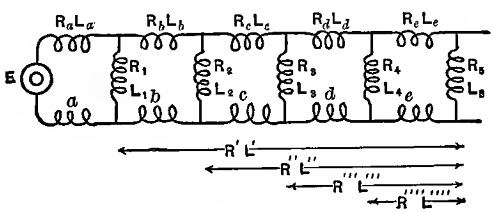


FIG. 73.—PROBLEM XII.

ance and self-induction $R_1 L_1$, $R_2 L_2$, $R_3 L_3$, etc., respectively. The resistance and self-induction of the mains are

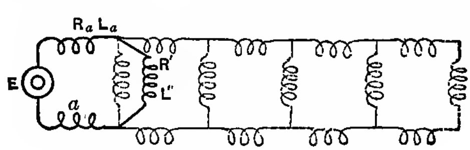


FIG. 74.—PROBLEM XII.

R_a and L_a for the portion a, R_b and L_b for the portion b, between circuits 1 and 2, R_c and L_c for the portion c, etc.

The equivalent resistance and self-induction of circuit one and that portion of the system beyond circuit one—namely, b, c, d, etc., and 2, 3, 4, etc.—is R' L'; for circuit two and the portion of the system beyond, the equivalent resistance

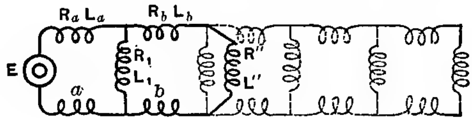


FIG. 75.—PROBLEM XII.

and self-induction are R'' and L''; for circuits three and beyond they are R''' and L'''; etc. These values of the equivalent resistance and self-induction are computed by successive applications of the formulæ (352) and (353), beginning at the most distant end of the system. The equivalent resistance R'''' and self-induction L'''' are found by adding R_5 and L_5 to R_e and L_e, respectively, and finding the equivalent resistance and self-induction when combined in parallel with circuit four. R''' and L''' are found by adding R'''' and L'''' to R_d and L_d and finding the equivalent resistance and self-induction of this when combined in parallel with circuit three. R'' and L'', and R' and L', are similarly found.

Let us now replace by a simple circuit with resistance and self-induction R' and L' that part of the system to which it is equivalent. The system then reduces to a series circuit (Fig. 74), and its solution is obtained by the method for series circuits, PROBLEM III. The complete solution for this problem is given in Fig. 76.

Draw $\overline{OA} = \overline{E}$. On $\overline{OA}$ erect the right triangle OBA so that $\tan AOB = \dfrac{L_a + L'}{R_a + R'}\,\omega$. Find the point C such that

$$\overline{OC} : \overline{CB} :: R_a : R'.$$

Draw $\overline{DC}$ perpendicular to $\overline{OB}$, and complete the triangle OCD so that $\tan DOC = \frac{L_a}{R_a}\omega$. Then E_a represents the

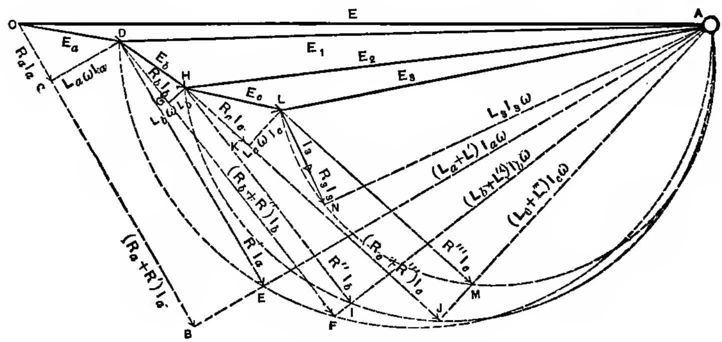


FIG. 76.—PROBLEM XII.

impressed E. M. F. of the portion a of the circuit, and E_1 the impressed E. M. F. of the remaining portion.

$$\tan ADE = \frac{L'}{R'}\omega.$$

Now let us take the system as originally shown in Fig. 73, and replace by a simple circuit with resistance R'' and self-induction L'' that part of the system to which it is equivalent. The system then reduces to the form shown in Fig. 75. The construction of Fig. 76 is continued as before.

On $\overline{DA}$, which represents E_1, the E. M. F. impressed at the terminals of the two parallel circuits, draw the right triangle DFA so that

$$\tan ADF = \frac{L_b + L''}{R_b + R''}\omega.$$

Divide $\overline{DF}$ at G so that

$$\overline{DG} : \overline{GF} :: R_b : R''.$$

Construct the right triangle HDG so that

$$\tan HDG = \frac{L_b}{R_b}\,\omega.$$

Then E_b is the E. M. F. impressed in the portion b of the circuit, and E_2 that impressed on the part of the circuit beyond b.

Repeated applications of this method of construction finally give the complete solution of the problem, and we have E_1, E_2, E_3, etc., as the E. M. F.'s impressed on the circuits 1, 2, 3, etc.; and E_a, E_b, E_c, etc., as the E. M. F.'s impressed on the portions a, b, c, etc.

Knowing the impressed E. M. F. on any simple portion of the circuit, a triangle of E. M. F.'s can be drawn and the current obtained. The E. M. F. triangles on E_a, E_b, and E_c are already drawn and the effective E. M. F.'s, $R_a I_a$, $R_b I_b$, $R_c I_c$, found. The current is found by dividing the effective E. M. F. by the resistance. In a similar way the current in each of the branch circuits 1, 2, 3, etc., may be found. For instance, on E_3 the E. M. F. triangle OLN is drawn. The effective E. M. F, LN, divided by the resistance gives the current, I_3.

The solution of this problem by entirely graphical methods could be gone through with, as in some of the previous problems, and likewise the problem of the same arrangement of circuits with the current in some portion of the circuit given.

CHAPTER XVII.

PROBLEMS WITH CIRCUITS CONTAINING RESISTANCE AND SELF INDUCTION. MORE THAN ONE SOURCE OF ELECTROMOTIVE FORCE.

Problem XIII. Electromotive Forces in Series.

SUPPOSE that in different parts of a single circuit there are two sources of harmonic E. M. F. It is required to find the current which flows and the various falls of potential in the different parts of the circuit.

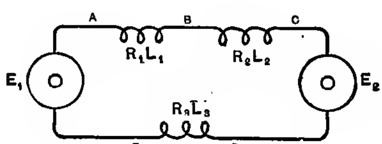


FIG. 77.—PROBLEM XIII.

Let the circuit be that represented in Fig. 77, where E_1 and E_2 are two different sources of harmonic E. M. F of the same period. Draw the lines $\overline{OA}$ and $\overline{OB}$, Fig. 78, to represent the E. M. F.'s E_1 and E_2 respectively.

The total E. M. F. acting in the circuit is the geometric sum of $\overline{OA}$ and $\overline{OB}$, that is, the diagonal $\overline{OC}$ (see page 213).

Regarding $\overline{OC}$ as the impressed E. M. F. in a single circuit, whose resistance is ΣR, and self-induction ΣL, we may

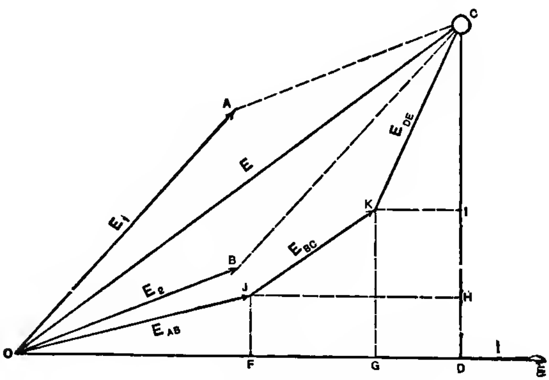


FIG. 78.—PROBLEM XIII.

construct the triangle of E. M. F.'s and thus find the current. Make the angle $C\,O\,D$ equal to $\tan^{-1} \dfrac{\Sigma\, L\, \omega}{\Sigma\, R}$. Then $\overline{OD}$ equals $I\,\Sigma\, R$, and $\overline{DC}$ equals $I\,\omega\,\Sigma\, L$. Dividing $\overline{OD}$ by $\Sigma\, R$, we obtain the current I, or $\overline{OE}$. To obtain the various falls of potential between the points $A\,B$, $B\,C$, and $E\,D$ (Fig. 77), divide $\overline{OD}$ at F and G into parts proportional to R_1, R_2, and R_3, and $\overline{DC}$ at H and I into parts proportional to L_1, L_2, and L_3. This determines the points J and K and thus gives the falls of potential $\overline{OJ}$, $\overline{JK}$, and $\overline{KC}$ for eacn part of the circuit.

Problem XIV. Direction of Rotation of E. M. F. Vectors.

When two harmonic E. M. F.'s of the same period are connected in series, the question may arise whether it may not happen that the vectors representing the two E. M. F.'s revolve in opposite directions. It is evident that, if they should revolve in opposite directions, the resultant

at any instant, instead of lying on a circle, lies upon an ellipse (Fig. 79). Here $\overline{OB}$ is an E. M. F. vector revolving

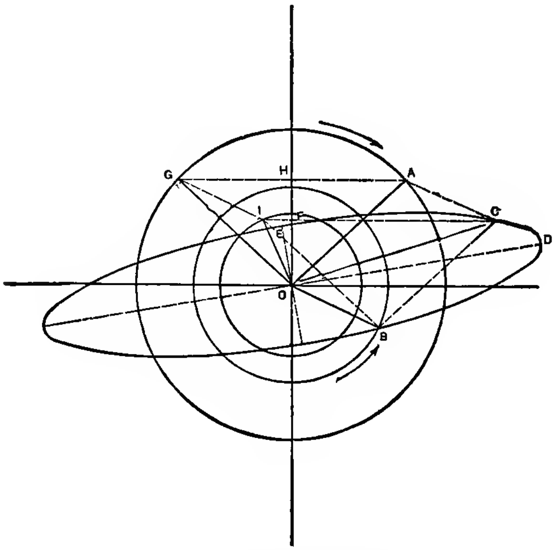


FIG. 79.—PROBLEM XIV.

counter-clockwise, and $\overline{OA}$ one revolving with the same angular velocity in the opposite direction. The resultant $\overline{OC}$ must always lie upon the ellipse. The major axis has a fixed direction $\overline{OD}$ which bisects the angle between $\overline{OA}$ and $\overline{OB}$. The magnitudes of the semi-major and the semi-minor axes are equal, respectively, to the arithmetical sum and the arithmetical difference of the vectors $\overline{OA}$ and $\overline{OB}$.

If, instead of drawing $\overline{OA}$ in the direction indicated, we had drawn it in the position $\overline{OG}$ (making the angle $G\,O\,H$ equal to $A\,O\,H$), and caused it to revolve counter-clockwise in the same direction as $\overline{OB}$, the projections, $\overline{OH}$, of $\overline{OA}$ or $\overline{OG}$ would be the same at every moment. Consequently the vector $\overline{OG}$ revolving counter-clockwise represents the same E. M. F. at every moment as the vector $\overline{OA}$ revolving clockwise, and may therefore be substituted

for it. But the resultant of $\overline{OG}$ and $\overline{OB}$ gives $\overline{OI}$, whose locus is a circle. Thus the projection of $\overline{OI}$ is the same as the projection of $\overline{OC}$, and the ellipse may therefore be replaced by the circle.

It is never necessary, therefore, to consider vectors revolving in opposite directions, for a vector revolving in one direction can always be replaced by a vector revolving in the opposite direction.

Problem XV. Electromotive Forces in Parallel.

Suppose that in each branch of a divided circuit, such as that represented in Fig. 80, there is a source of harmonic E. M. F., and that all these E. M. F.'s have the same period; it is required to find the currents in the branches.

The currents may be found by making use of the general principle* that, if the currents due to each E. M. F. acting separately can be found, the current which flows when all the E. M. F.'s are acting together is the geometrical sum of all these partial currents.

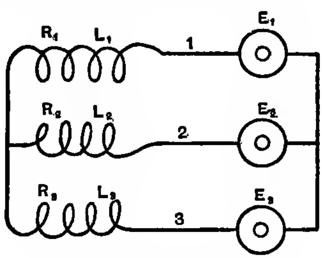


FIG. 80.—PROBLEM XV.

To find the currents due to all the E. M. F.'s acting together we may then proceed by regarding each branch, 1, 2, and 3, in turn, as the main line in which there is the impressed E. M. F., and the other branches as a divided circuit.

* See Mascart and Joubert's Electricity and Magnetism, Vol. 1, Art. 202.

Then, considering E_1 to be the only E. M. F. acting, the problem of finding the partial currents I_1', I_2', and I_3' is readily solved by the methods already given. Next, considering E_2 as acting alone, we may find the partial currents I_1'', I_2'', and I_3'', and finally we find I_1''', I_2''', and I_3''', due to E_3 acting alone.

The actual currents in the branches I_1, I_2, and I_3 when all the E. M. F.'s act together, by the principle just stated, must be equal to the geometrical sum of the partial currents; that is,

$I_1 =$ geometrical sum of I_1', I_2', and I_3';
$I_2 =$ " " " I_1'', I_2'', and I_3'';
$I_3 =$ " " " I_1''', I_2''', and I_3'''.

Problem XVI. Electromotive Forces Having Different Periods.

Let there be two impressed harmonic E. M. F.'s in series having periods which bear a ratio of three to one. It is required to find the resultant impressed E. M. F. and the current that flows in the circuit.

In Fig. 81 let $\overline{OA}$ represent maximum value of E_1, and $\overline{OB}$ that of E_2, they being in the ratio of one to two. As $\overline{OA}$ revolves around its circle three times as fast as $\overline{OB}$, $\overline{OA}$ arrives at $\overline{OC}$ when $\overline{OB}$ arrives at $\overline{OD}$, and the resultant $\overline{OF}$ traverses the curve $EFGH$. If the projection of the resultant vector $\overline{OF}$ is taken upon the axis $\overline{OY}$ at equal intervals of time, we may thus plot the curve of resultant E. M. F., Fig. 82. This E. M. F. curve is the plot of the equation

$$e = E_1 \sin 3\,\omega t + E_2 \sin \omega t.$$

The curve is composed of two simple harmonic components.

To find the current which this resultant E. M. F. causes to flow, it is only necessary to find the currents which each

component E. M. F. acting separately would cause, and then add these together geometrically. If there is self-induction in the circuit, the tangent of the angle of lag of the com-

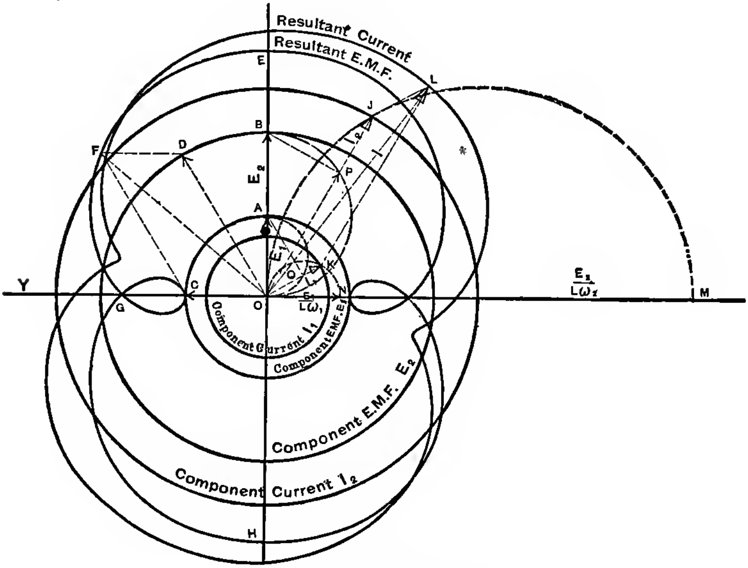


FIG. 81.—PROBLEM XVI.

ponent currents behind their respective E. M. F.'s is $\frac{L\,\omega}{R}$. Let $O\,P\,B$ be the E. M. F. triangle upon E_2, and $\overline{O\,J}$ the current I_2. J must lie upon the semi-circle $O\,J\,M$, whose diameter is $\frac{E_2}{L\,\omega_2}$ (see PROBLEM I.). The angle of lag, $A\ O\ Q$, of the component current due to the E. M. F., E_1, is now determined, since its tangent is three times the tangent of $B\ O\ P$, thus, $\frac{L\,\omega_1}{R} = \frac{3\,L\,\omega_2}{R}$. Also the current $\overline{O\,K}$, or I_1, due to the E. M. F. $\overline{O\,A}$, or E_1, is now determined, since K must lie upon a semi-circle $O\,K\,N$ whose diameter $\overline{O\,N}$ equals $\frac{1}{6}$ of $\overline{O\,M}$. For $\overline{O\,N} = \frac{E_1}{L\,\omega_1} = \frac{E_1}{3\,L\,\omega_2}$; and $\overline{O\,M} =$

$\frac{E_2}{L\,\omega_2} = \frac{2\,E_1}{L\,\omega_2}$, and thus $\overline{OM} = 6\ \overline{ON}$. The resultant of $\overline{OK}$ and $\overline{OJ}$ gives $\overline{OL}$. and this vector always follows the curve marked "Resultant Current." If the projection of $\overline{OL}$ upon the axis $\overline{OY}$ is taken at regular intervals as L moves around its curve, we may obtain the current curve Fig. 82.

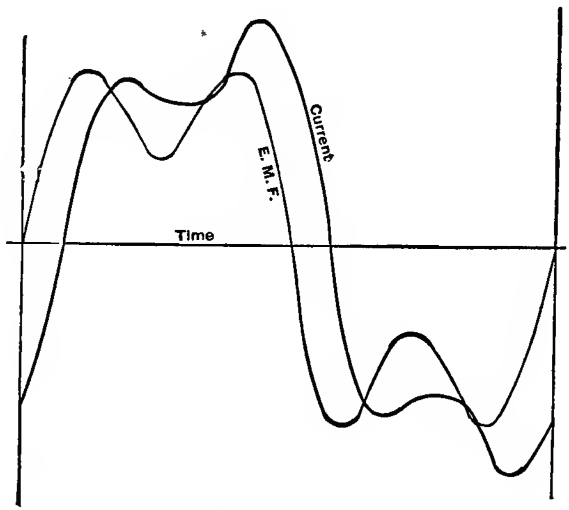


FIG. 82.—PROBLEM XVI.

This current curve is composed of two simple harmonic curves each due to a simple harmonic E. M. F., but the two component current curves lag behind their respective component E. M. F. curves by different angles. For this reason the resultant current curve is not symmetrical with the resultant E. M. F. curve.

CHAPTER XVIII.

INTRODUCTORY TO CIRCUITS CONTAINING RESISTANCE AND CAPACITY.

CONTENTS:—Problems with R and C analytically and graphically analogous to problems with R and L. Triangle of E. M. F.'s for a single circuit containing resistance and capacity. Impressed E. M. F. Effective E. M. F. Condenser E. M. F. Direction shown from differential equations. Graphical representation. Two methods used. First method (the one used throughout this book), employing E. M. F. necessary to overcome that of condenser. Second method, employing E. M. F. of condenser. Further identification of analytical and graphical relations. Mechanical analogue.

WHEN Chapter III., giving the analytical treatment of circuits containing resistance and self-induction, is compared with Chapter V., which gives the corresponding analytical treatment of circuits containing resistance and capacity, the similarity leads us to infer that the graphical solutions of problems will be very analogous in the two cases. Although the analogy is very close, which fact makes it much easier to follow the solutions for resistance and capacity and is a great help, yet, in many respects, the contrast is so marked that it is considered advisable, in discussing problems with circuits containing resistance and capacity, to give the solutions for the same arrangement of circuits as those which have been given for circuits containing resistance and self-induction in the previous pages, in order that the points of similarity and difference may be clearly understood.

Triangle of Electromotive Forces for a Single Circuit Containing Resistance and Capacity.

In Chapter V., in which circuits containing resistance and capacity were analytically treated, it was shown [equation (78)] that when the impressed E. M. F. is harmonic, that is,

$$e = E \sin \omega t,$$

the resulting current which flows is also harmonic and is

$$i = \frac{E}{\sqrt{R^2 + \frac{1}{C^2 \omega^2}}} \sin\left[\omega t + \tan^{-1} \frac{1}{CR\omega}\right]. \tag{78}$$

The charge of the condenser is likewise harmonic and is [equation (79)]

$$q = \frac{E}{\omega\sqrt{R^2 + \frac{1}{C^2 \omega^2}}} \sin\left[\omega t + \tan^{-1} \frac{1}{CR\omega} - 90^\circ\right].$$

These equations for the current and charge were derived from the differential equation of electromotive forces which may be written in any of the forms [see (55)]

$$e = Ri + \frac{\int i\,dt}{C},$$

$$e = Ri + \frac{q}{C},$$

$$\text{or} \qquad \frac{de}{dt} = R\frac{di}{dt} + \frac{i}{C}.$$

Here e is the instantaneous value of the impressed E. M. F. of the source; Ri is that part *necessary to overcome* the ohmic resistance; and $\frac{\int i\,dt}{C}$ or $\frac{q}{C}$ the E. M. F. *necessary to overcome* the E. M. F. of the condenser.

Let the vector, $\overline{OA}$, Fig. 83, represent the harmonic impressed E. M. F. of the source. Then, by equation (78),

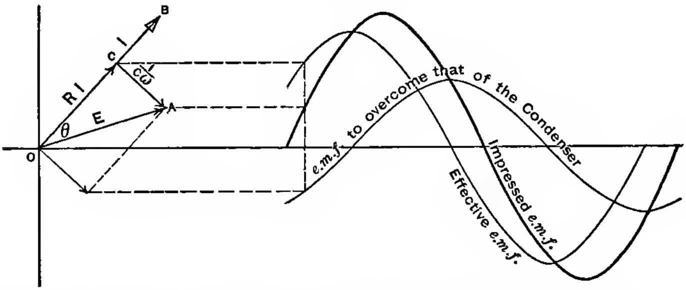


FIG. 83.—TRIANGLE OF ELECTROMOTIVE FORCES.
FIRST METHOD—THE ONE USED THROUGHOUT THIS BOOK—EMPLOYING E. M. F. TO OVERCOME THAT OF THE CONDENSER.

we see that the current must be represented by a vector, $\overline{OB}$, in advance of $\overline{OA}$ by an angle θ, or $\tan^{-1}\frac{1}{CR\omega}$. The effective E. M. F., being equal to RI, has the same direction as the current and must be represented by a vector $\overline{OC}$ equal to the current vector, $\overline{OB}$, multiplied by R. The E. M. F. to overcome that of the condenser, having the instantaneous value $\frac{q}{C}$, is at right angles to the current, and must therefore be represented by the vector $\overline{CA}$ perpendicular to $\overline{OB}$.

It may be shown that this E. M. F. $\frac{q}{C}$ is at right angles to the current by the preceding equations, thus:

$$\frac{q}{C}=\frac{\int i\,d\,t}{C}=\frac{E}{C\,\omega\sqrt{R^2+\frac{1}{C^2\,\omega^2}}}$$

$$\sin\left[\omega\,t+\tan^{-1}\frac{1}{C\,R\,\omega}-90^\circ\right].$$

To simplify this expression substitute

$$I = \frac{E}{\sqrt{R^2 + \frac{1}{C^2 \omega^2}}},$$

and $$\theta = \tan^{-1} \frac{1}{C R \omega}.$$

The equation then becomes

(354) $$\frac{q}{C} = \frac{I}{C \omega} \sin \left[\omega t + \theta - 90^\circ \right].$$

This equation shows that the E. M. F., $\frac{q}{C}$, *to overcome that* of the condenser is ninety degrees *behind* the current, and that the maximum value of this E. M. F. is $\frac{I}{C \omega}$. The vector (Fig. 83), whose length is $\frac{I}{C \omega}$, ninety degrees behind the current, $\overline{OB}$, therefore represents the E. M. F. *to overcome* that of the condenser.

The E. M. F. *of* the condenser is equal and opposite to that which is *necessary to overcome* it, and is consequently ninety degrees in *advance* of the current represented by the vector, $\overline{AC}$, Fig. 84.

The Method to be Used in the Graphical Solutions of Problems for Circuits Containing Resistance and Capacity.

In the graphical treatment of problems with circuits containing resistance and capacity, just as was the case with circuits containing resistance and self-induction, there are two methods of drawing, each equally correct, which will, if followed throughout, give identically the same results. These two methods arise according to whether the E. M. F. *necessary to overcome* the E. M. F. of the condenser is con-

sidered, or the E. M. F. *of* the condenser. The first method is illustrated by Fig. 83; the second by Fig. 84.

In order that uniformity may exist throughout all the diagrams which represent cases where both self-induction and capacity are considered in circuit, since the method of

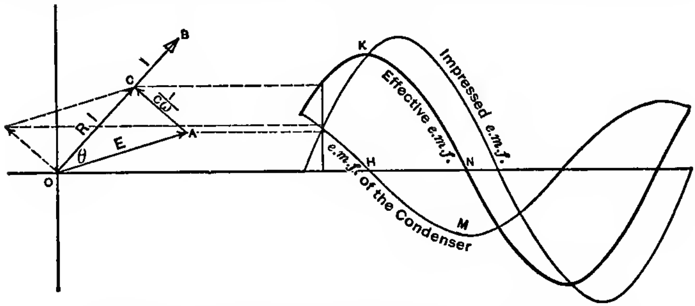


FIG. 84.—TRIANGLE OF ELECTROMOTIVE FORCES.
SECOND METHOD, EMPLOYING E. M. F. OF CONDENSER.

drawing was adopted which considered the E. M. F. necessary to overcome the self-induction, here we are obliged to adopt that method which employs the E. M. F. *necessary to overcome* the E. M. F. of the condenser, as represented in Fig. 83.

That the construction of the figures fulfils the conditions expressed by the current equation (78) may be shown again by a further comparison of the relations. Thus in Fig. 83 or 84 it is evident that

$$\tan A\,O\,C = \frac{A\,C}{O\,C} = \frac{\frac{I}{C\,\omega}}{R\,I} = \frac{1}{C\,R\,\omega} = \tan\theta,$$

and this corresponds to the angle of advance. Again, the impressed E. M. F., $\overline{O\,A}$, being the hypotenuse of the triangle $O\,A\,C$, is equal to the square root of the sum of the squares of the two sides, and, therefore,

$$\overline{O\,A} = \sqrt{\overline{O\,C}^2 + \overline{C\,A}^2};$$

that is, $$E = \sqrt{R^2 I^2 + \frac{I^2}{C^2 \omega^2}} = I \sqrt{R^2 + \frac{1}{C^2 \omega^2}},$$

and $$I = \frac{E}{\sqrt{R^2 + \frac{1}{C^2 \omega^2}}}.$$

This is seen to correspond to the maximum value of the current given in equation (78).

MECHANICAL ANALOGUE.

That the E. M. F. of the condenser is at right angles to the current may, perhaps, be best understood by the physical conception of the part played by the condenser in a circuit. A good mechanical analogue of the condenser is an air-chamber, as represented in Fig. 85, in which the air is first compressed and then expanded. The piston P moves back and forth, with an harmonic motion, we will say, first compressing and then expanding the air in the chamber. When at its central position, the air is at the atmospheric pressure. The current may be represented by the motion of the piston, or of the water in the tube which transmits the pressure to the air-chamber. The charge of the condenser may be represented by the volume of water which enters or leaves the air-chamber, the charge being taken as zero when the piston is at its central position, that is, when the air is at the atmospheric pressure. Considering the moment when the piston is in the central position P, moving upward, the charge is zero, and the current is a maximum, as here the piston moves with its maximum velocity. The cor-

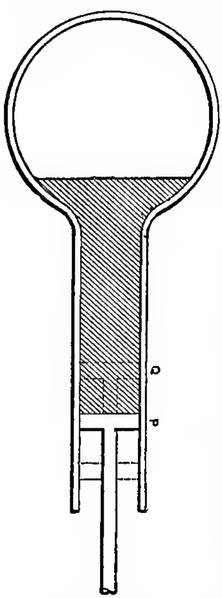


FIG. 85.—MECHANICAL ANALOGUE OF A CONDENSER.

responding points on the curves, Fig. 84, are *H* and *K*; that is, the positive current is represented by the upward motion of the piston. When the piston arrives at *Q*, the upper end of the stroke, the current is zero and is represented by the point *N* on the curve. The charge is here a positive maximum, and during the previous quarter of the stroke the compressed air has exerted an outward pressure, corresponding to the E. M. F. *of* the condenser, opposed to the current. This pressure reaches a negative maximum, together with the charge, when the current is zero. This corresponds to the point *M* on the curve. During the return of the piston to the central position, both the current and the pressure are in the same negative direction until the current becomes a negative maximum, at the central position, where the pressure becomes zero and then changes sign. This example shows how the pressure exerted by the air, corresponding to the E. M. F. *of* the condenser, is just ninety degrees in advance of the current. The pressure which must be exerted upon the piston *to overcome* the pressure of the air chamber, corresponding to the E. M. F. necessary to overcome that of the condenser, is evidently equal and opposite to the pressure *of* the air-chamber, and lags, therefore, ninety degrees behind the current. As before explained, Fig. 83 represents the manner of drawing when the E. M. F. *necessary to overcome* that of the condenser is considered, and Fig. 84 when the E. M. F. *of* the condenser is considered.

CHAPTER XIX.

PROBLEMS WITH CIRCUITS CONTAINING RESISTANCE AND CAPACITY.

Problem XVII. Effects of the Variation of the Constants *R* and *C* in a Series Circuit.

The Resistance Varied.

When the ohmic resistance is varied in a circuit containing only resistance and capacity, the current is changed

and it is of interest to investigate just how it changes both in magnitude and in direction. The triangle $O A C$, Fig. 86, represents the triangle of E. M. F.'s for the circuit

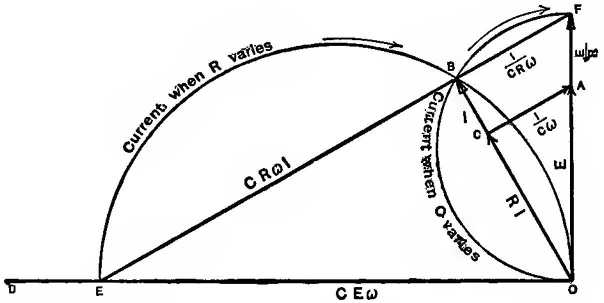


FIG. 86.—VARIATION OF RESISTANCE AND CAPACITY IN A SERIES CIRCUIT. PROBLEM XVII.

when the resistance is R. The current $\overline{O B}$ is equal to $\overline{O C}$ divided by R. Draw a line $\overline{O D}$, of indefinite length, perpendicular to the E. M. F. $\overline{O A}$ in the direction of advance. The angle $D O C$ is the complement of $A O C$, and is, therefore, $\tan^{-1} C R \omega$. Draw $\overline{B E}$ perpendicular to $\overline{O B}$ and let it meet $\overline{O D}$ at E. The line $\overline{B E}$ then equals $C R \omega I$; for, $\overline{O B}$ equals I, and $\tan E O B$ equals $C R \omega$.

It can be shown that the hypotenuse $\overline{O E}$ of this triangle is equal to $C E \omega$, and is therefore a constant entirely independent of any variation in the current I, or resistance R. Taking the square root of the sum of the squares of the sides $\overline{O B}$ and $\overline{B E}$, we obtain

$$\overline{O E} = \sqrt{\overline{O B}^2 + \overline{B E}^2} = I \sqrt{1 + C^2 R^2 \omega^2}.$$

Substituting for I its value $\dfrac{E}{\sqrt{R^2 + \dfrac{1}{C^2 \omega^2}}}$, we obtain

$$I \sqrt{1 + C^2 R^2 \omega^2} = C E \omega,$$

and, therefore, $$\overline{O E} = C E \omega.$$

Now since the side $\overline{OB}$ of the right triangle OBE always represents the current I, and the hypotenuse $\overline{OE}$ is independent of current or resistance, it follows that the current is always represented by a vector $\overline{OB}$ inscribed in a semicircle OBE for any possible variation in the resistance. In the figure the arrow indicates the direction of variation as the resistance increases.

In the limiting cases when R is infinite or zero, we see by this figure the limiting values of the current. When R is infinite, the current is evidently zero. When R approaches zero, $\overline{OB}$ approaches $\overline{OE}$, and in the limit the current becomes

$$I = C E \omega.$$

When the circuit contains no ohmic resistance, we see, first, that the impressed E. M. F. is equal to $\frac{I}{C\omega}$, the E. M. F. of the condenser; and, second, that the current is 90° in advance of the impressed E. M. F. These relations, here geometrically shown, are analytically expressed in equation (354).

The Capacity of the Condenser Varied.

Suppose that the capacity of the condenser in the circuit is varied while the resistance remains the same; we wish to find how the current changes.

In the same figure, 86, prolong the line $\overline{EB}$ until it meets the impressed E. M. F. $\overline{OA}$ prolonged at F. Then $\overline{BF}$ equals $\frac{I}{CR\omega}$, since $\tan BOF$ equals $\frac{1}{CR\omega}$.

The hypotenuse $\overline{OF}$ is, therefore,

$$\overline{OF} = \sqrt{\overline{OB}^2 + \overline{BF}^2} = I\sqrt{1 + \frac{1}{C^2 R^2 \omega^2}}.$$

From the value for I in (82) it follows that

$$I\sqrt{1+\frac{1}{C^2 R^2 \omega^2}}=\frac{E}{R}.$$

Hence, $$\overline{OF}=\frac{E}{R}.$$

Since the hypotenuse $\overline{OF}$ is independent of the current I or the capacity C, and is a constant for any variation in C, it follows that the current is always represented by a vector OB inscribed in the semi-circle OBF for any possible value of the capacity. In the figure the arrow indicates the direction of variation as the capacity increases.

In the limiting cases when C is zero or infinite, we see from the figure the value of the current. When C approaches zero, the current evidently approaches zero. When C approaches infinity (which is equivalent to having no condenser in the circuit), the current vector $\overline{OB}$ approaches $\overline{OF}$, and, in the limit, $I=\frac{E}{R}$, and the current follows Ohm's law.

That the construction of Fig. 86 is consistent with the equations is further shown from the following relations.

$$\text{(355)}\quad \overline{EF}^2=(\overline{EB}+\overline{BF})^2=\left\{CR\omega I+\frac{I}{CR\omega}\right\}^2$$

$$=\frac{I^2}{C^2 R^2 \omega^2}\{1+C^2 R^2\omega^2\}^2.$$

$$\text{(356)}\quad \overline{OE}^2+\overline{OF}^2=C^2E^2\omega^2+\frac{E^2}{R^2}=\frac{E^2}{R^2}(1+C^2R^2\omega^2).$$

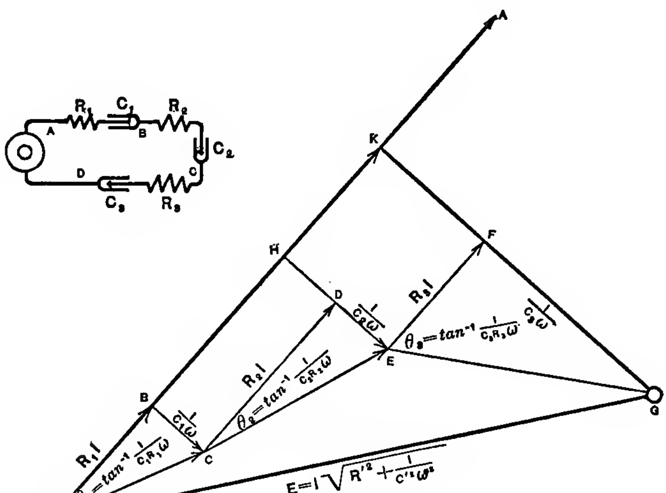
Equating (355) and (356), we find

$$\frac{I}{C^2\ \omega^2}(1+C^2R^2\omega^2)=E^2,\quad\text{or}\quad I=\frac{E}{\sqrt{R^2+\frac{1}{C^2\omega^2}}},$$

a result which is identical with that analytically expressed in equation (82).

Problem XVIII. Series Circuit. Current Given.

Let there be a circuit, Fig. 87, having in series n different resistances R_1, R_2, etc., and n condensers with capacities C_1, C_2, etc. It is required to find the impressed E. M. F. necessary to cause a current I to flow. In Fig. 88, make $\overline{O\,A}$ equal to the current flowing. Multiply this



FIGS. 87 AND 88.—PROBLEM XVIII. AND PROBLEM XIX.

by R_1, and lay off $\overline{O\,B}$ equal to $R_1\,I$, which is, then, the effective E. M. F. to overcome the resistance R_1. Draw $\overline{B\,C}$ perpendicular to $\overline{O\,A}$ in the negative direction, and make the angle $B\;O\;C = \theta_1 = \tan^{-1}\frac{1}{C_1\,R_1\,\omega}$. Then $B\;O\;C$ is the triangle of E. M. F.'s for that part of the circuit between A and B, Fig. 87.

The construction of the figure is similar to that of Fig.

54, in PROBLEM II., but differs from Fig. 54 in the fact that the E. M. F. triangles in the present construction are so drawn that the various currents are in advance of their respective electromotive forces. The triangles $C D E$, etc., are drawn and the construction completed similar to the corresponding case, PROBLEM II., of a series circuit with self-induction. We thus find the impressed E. M. F. to be $\overline{O G}$.

EQUIVALENT RESISTANCE AND EQUIVALENT CAPACITY IN SERIES.

Suppose that we replace all the resistances in Fig. 87 by a single resistance, and all the condensers by a single one; it is required to find that resistance and capacity which will allow the same current to flow.

It is evident that if $\overline{O K}$, Fig. 88, is $R'I$, and $\overline{K G}$ is $\frac{I}{C' \omega}$, where R' and C' represent, respectively, the equivalent resistance and equivalent capacity, the same current $\overline{O A}$ will flow. But $\overline{O K} = I \Sigma R$, and $\overline{K G} = \frac{I}{\omega} \Sigma \frac{1}{C}$. It therefore follows that $R' = \Sigma R$, and $\frac{1}{C'} = \Sigma \frac{1}{C}$. We may write it $C' = \frac{1}{\Sigma \frac{1}{C}}$ and have the equivalent capacity equal to the reciprocal of the sum of the reciprocals of each separate capacity.

Problem XIX. Series Circuit. Impressed E. M. F. Given.

The circuit being the same as in Fig. 87 in the previous problem, it is required to find the current and the fall of potential through each of the various parts of the circuit when the impressed E. M. F. is given. From the remarks on equivalent resistance and capacity immediately preceding, it is

evident that the same current will flow if these equivalents are substituted for the separate resistances and capacities. The triangle $O\,K\,G$ may now be drawn and the current found. From this point we may proceed as in the preceding problem to find the various falls of potential $\overline{O\,C}$, $\overline{C\,E}$, and $\overline{E\,G}$.

Problem XX. Divided Circuit. Two Branches. Impressed E. M. F. Given.

Let us consider the problem of a divided circuit having two branches in parallel, as indicated in Fig. 89. Each

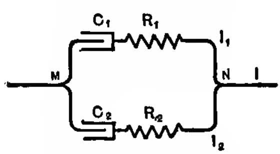


FIG. 89.—PROBLEM XX.

branch contains resistance and capacity, and there is an impressed E. M. F., E, between the terminals M and N; it is required to find the main current, I, and the currents I_1 and I_2 in the branches.

This problem corresponds very closely to PROBLEM IV., in which case the branches contain self-induction instead of capacity. Fig. 90 represents the solution of the present problem, and corresponds to Fig. 56, PROBLEM IV. The difference is that the E. M. F. triangles $O\,B\,A$ and $O\,C\,A$, Fig. 90, lie on the positive or advance side of the impressed E. M. F. $O\,A$, instead of on the negative side as in Fig. 56. Otherwise the construction by which we obtain the two currents $\overline{O\,D}$ and $\overline{O\,E}$, and the resultant main current, $\overline{O\,F}$, is identical with that in PROBLEM IV.

EQUIVALENT RESISTANCE AND CAPACITY.

Suppose that, instead of the two parallel branches just considered, a single circuit be substituted for them whose resistance, R', and capacity, C', is such that the same cur-

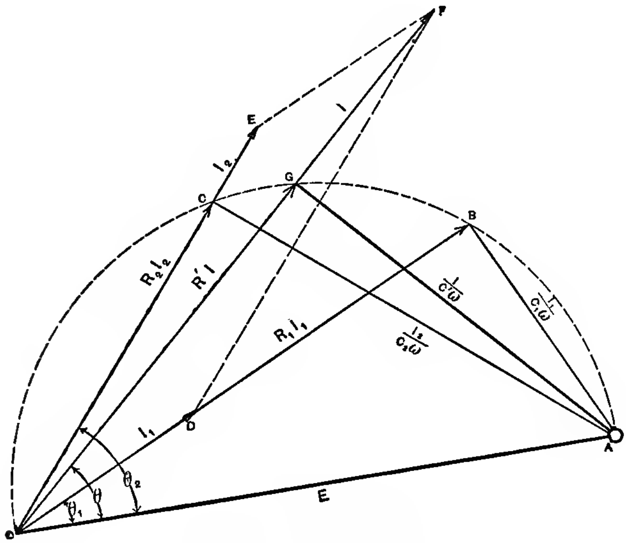


FIG. 90.—PROBLEM XX.

rent as before will flow in the main line. The triangle of E. M. F.'s for this equivalent circuit must be $O\,G\,A$, Fig. 90, since the impressed E. M. F. is $\overline{O\,A}$, and the effective E. M. F. is $\overline{O\,G}$ in the direction of the current, and the E. M. F. $\overline{G\,A}$, to overcome that of the condenser, is at right angles to the current. We may write, therefore, $R'I$ for $\overline{O\,G}$, and $\dfrac{I}{C'\omega}$ for $\overline{G\,A}$. This equivalent resistance and capacity may be expressed in terms of the resistances and capacities of the branches, but the determination of these values will be deferred until after the discussion of the following problem.

Problem XXI. Divided Circuit. Any Number of Branches. Impressed E. M. F. Given.

Let the divided circuit, MN, Fig. 91, have n branches in parallel, each containing resistance and capacity, with an impressed E. M. F., E, between the terminals. It is required to find the main current I.

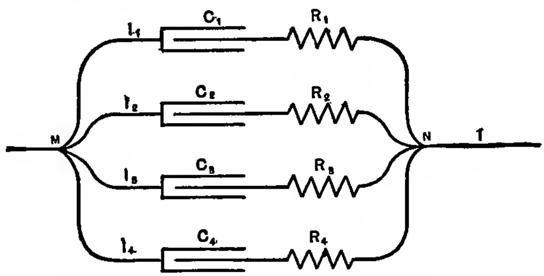


Fig. 91.—Problem XXI. and Problem XXII.

The construction of Fig. 92 is similar to that of Fig. 58, in Problem V., except that the E. M. F. triangles and all the branch currents are laid off in the direction of advance and not of lag. The main current $\overline{OL}$ is the geometrical resultant of all the branch currents $\overline{OF}$, $\overline{OG}$, $\overline{OH}$, and $\overline{OI}$, as before.

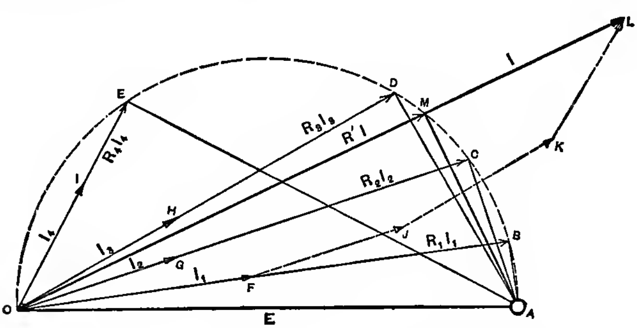


Fig. 92.—Problem XXI. and Problem XXII.

This diagram gives the complete solution of the problem of the divided circuit containing resistance and capacity. Here, too, as was the case with the divided circuit contain-

ing resistance and self-induction, it is evident that the maximum main current, I, is greater than any of the branch circuits.

EQUIVALENT RESISTANCE AND CAPACITY OF PARALLEL CIRCUITS.

In this case, as in the previous one, we may suppose a single circuit substituted for the parallel branches, having such a resistance, R', and capacity, C', that the current in the main line is not altered in magnitude or phase. The values of this equivalent resistance and capacity in terms of the resistances and capacities of the branches may be found by proceeding in the same way as was done to obtain the values of the equivalent resistance and self-induction of parallel circuits, PROBLEM V. Equations are formed by taking the projections of the currents first upon the line $\overline{OA}$, Fig. 92, and then upon a line perpendicular to $\overline{OA}$. In these equations, values for I, I_1, I_2, etc.; $\cos\theta$, $\cos\theta_1$, $\cos\theta_2$, etc.; $\sin\theta$, $\sin\theta_1$, $\sin\theta_2$, etc., obtained from the geometry of the figure, are substituted, and, after operations similar to those used in obtaining equivalent resistance and self-induction, the following expressions are obtained for the equivalent resistance and capacity of parallel circuits.

(356 a) $$R' = \frac{A}{A^2 + B^2\omega^2},$$

(356 b) and $$\frac{1}{C'\omega} = \frac{B\omega}{A^2 + B^2\omega^2},$$

where $$A = \sum \frac{R}{R^2 + \frac{1}{C^2\omega^2}},$$

and $$B\omega = \sum \frac{\frac{1}{C\omega}}{R^2 + \frac{1}{C^2\omega^2}} = \sum \frac{C\omega}{C^2R^2\omega^2 + 1}.$$

The main current is in advance of the impressed E. M. F. by an angle θ such that

$$\tan \theta = \frac{B\,\omega}{A}.$$

The complete proof of this was first given by the authors in the Philosophical Magazine for September, 1892. These results may be obtained from the general expressions for equivalent resistance, self-induction, and capacity which are discussed in PROBLEM XXXI.

Problem XXII. Divided Circuit. Current Given.

Suppose that we have a number of circuits, each with resistance and capacity, connected in parallel as in Fig. 91, and we know the value of the current I in the main line. We wish to find the current in the several branches. There are two solutions similar to the two given for the corresponding case of circuits with self-induction.

FIRST METHOD. ENTIRELY GRAPHICAL.

By assuming any value we please for the impressed E. M. F., E, we may solve as in the foregoing problem. The scale of the drawing must then be changed in the ratio of the given value of the main current, I, to the value of I thus obtained according to the assumed impressed E. M. F.

SECOND METHOD. BY USE OF EQUIVALENT RESISTANCE AND CAPACITY.

The problem may be otherwise solved by use of equivalent resistance and capacity of parallel circuits as given in (356 a) and (356 b). $\overline{OM}$, Fig. 92, is laid off equal to $R'I$.

The line $\overline{MA}$ is drawn perpendicular to $\overline{OM}$ and equal to $\frac{I}{C'\omega}$. The hypotenuse $\overline{OA}$ is the impressed E. M. F. The further construction is the same as in the foregoing problem. Upon $\overline{OA}$ the E. M. F. triangle for each branch is drawn and the current and angle of advance found.

Problem XXIII. Effects of the Variation of the Constants *R* and *C* in a Divided Circuit of Two Branches.

If we compare PROBLEMS I. and XVII., in which the discussion is given of the effects of the variation of the constants R and L, and R and C in series circuits, we see that

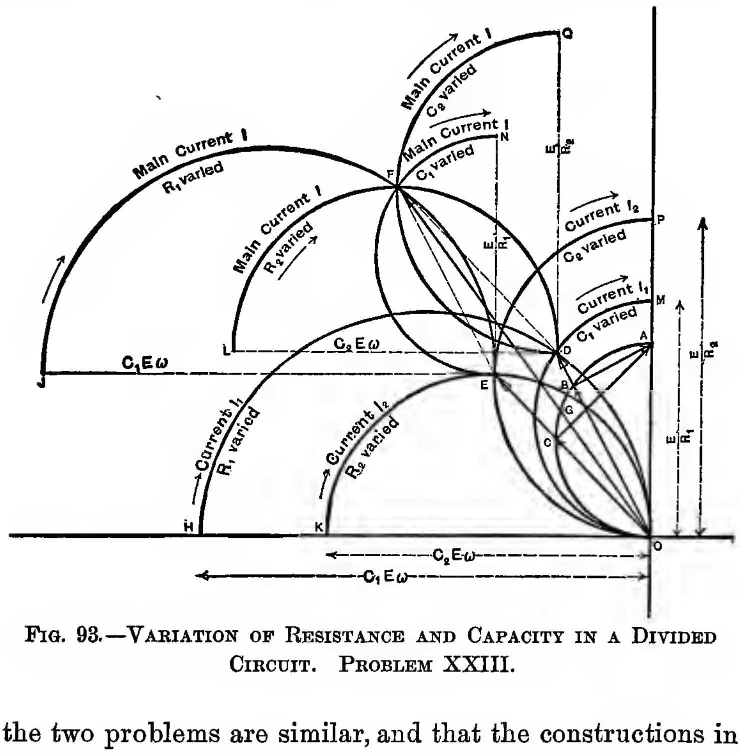


FIG. 93.—VARIATION OF RESISTANCE AND CAPACITY IN A DIVIDED CIRCUIT. PROBLEM XXIII.

the two problems are similar, and that the constructions in Figs. 52 and 86 are the same except for direction, the

former being in the direction of lag and the latter in the direction of advance. The present problem is similar to PROBLEM VII., in which the effect of the variation of R and L in a divided circuit is considered. The construction is given in Fig. 93, which explains itself, and is exactly similar to that given in Fig. 59, which was fully described in PROBLEM VII. The arrows in the figure indicate the direction of the change as the resistance or capacity increases.

Problem XXIV. Series and Parallel Circuits. Impressed E. M. F. Given. Solution by Use of Equivalent Resistance and Capacity.

Problems arising from combinations of series and parallel circuits with resistance and capacity are solved by the repeated application of the methods used for the foregoing simple problems in the same way as were solved the problems involving combinations of circuits with resistance and self-induction. Let us consider the case in which two systems of parallel circuits are joined in series, as in Fig. 94. The resistance and capacity of each branch and the

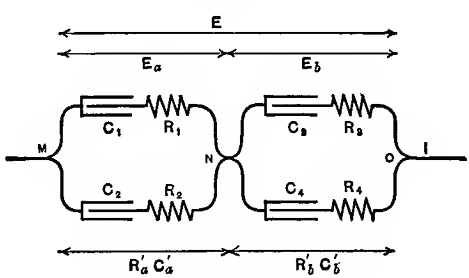


FIG. 94.—PROBLEM XXIV.

total impressed E. M. F. are given. It is required to find the current in the main line and branches. The problem is similar to PROBLEM VIII., and the solution given in Fig. 95

is obtained by a construction similar to Fig. 63. The equivalent resistance and capacity R_a' and C_a' between M and N, and R_b' and C_b' between N and O, are calculated according to (356 a) and (356 b). The impressed E. M. F.'s E_a and E_b are now found according to the method for series circuits, PROBLEM XIX. The part between M and N and

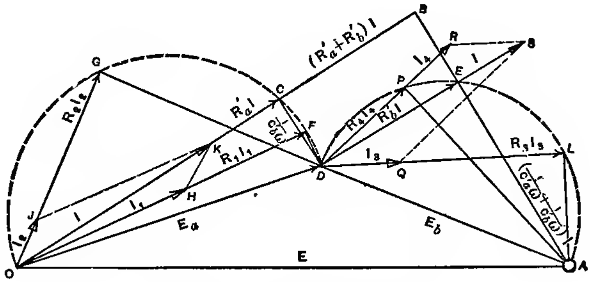


FIG. 95.—PROBLEM XXIV.

the part between N and O are now separately treated by the method of parallel circuits, PROBLEM XXI. The construction is shown clearly by the figure.

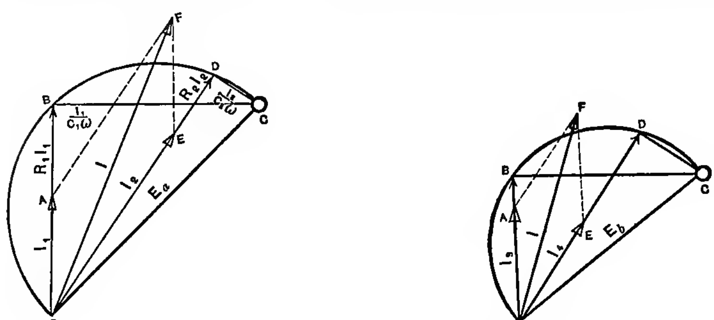
A more extended system of circuits in series and parallel is solved by the same methods.

Problem XXV. Series and Parallel Circuits. Current Given. Solution by Use of Equivalent Resistance and Capacity.

Let us suppose the same arrangement of circuits as that shown in Fig. 94, and that the main current, I, is given. It is required to find the current in each branch. The parts between M and N and between N and O may be separately solved according to the second method given in PROBLEM XXII. The solution of any number of circuits in series and parallel could be readily obtained by the same method.

Problem XXVI. Series and Parallel Circuits. Entirely Graphical Solution.

In the foregoing treatment of problems involving series and parallel combinations of circuits containing resistance and capacity it was necessary to find analytically the values of the equivalent resistance and capacity of each set of parallel circuits, and the solutions were, therefore, partly analytical and partly graphical. They may be obtained, as in the corresponding cases of combinations of circuits with resistance and self-induction (see PROBLEM XI.), by entirely graphical methods by assuming the value of the current in a particular branch or assuming its impressed E. M. F. After solving in this way, the values assumed and the scale of the diagrams must be altered to agree with the given conditions of the problem. Figs. 96, 97, 98, and 99 give the



FIGS. 96 AND 97.—PROBLEM XXVI.

construction for the entirely graphical solution of two parallel sets of circuits connected in series, as in Fig. 94. The method is to solve separately each parallel set of circuits by assuming some value for the impressed E. M. F. or for one of the branch currents. Figs. 96 and 97 give the construction for the solutions of the parts MN and NO, respectively, starting with assumed values for the branch currents I_1 and I_3. Fig. 97 is then magnified, as shown in

Fig. 98, until the main current I is the same size as in Fig. 96. The two figures, 96 and 98, are now combined in Fig. 99 so that $\overline{OF}$ is parallel to $\overline{O'F'}$, since each represents the

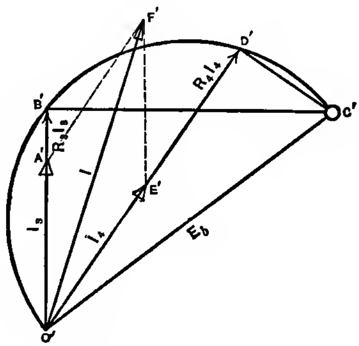


FIG. 98.—PROBLEM XXVI.

current I. We thus find E, the impressed E. M. F. which will cause the current I to flow. If the value of the impressed E. M. F. had been given, the scale of the diagram

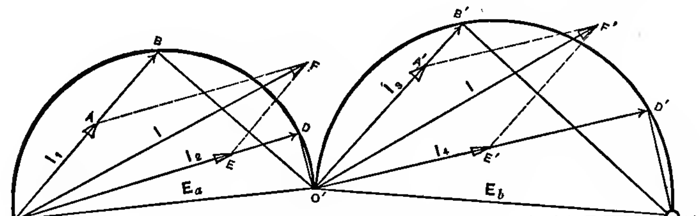


FIG. 99.—PROBLEM XXVI.

could be altered until E equaled the given value of the E. M. F. The figure would then give the value of the main and branch currents which flow when there is this given E. M. F.

Problem XXVII. Multiple-arc Arrangement.

Of the many arrangements in which circuits with resistance and capacity may be combined, let us consider, as a

further example, the arrangement in multiple arc, as shown in Fig. 100. The solution is obtained by dividing the

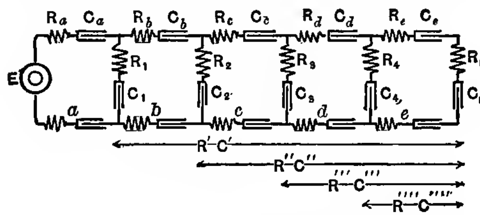


FIG. 100.—PROBLEM XXVII.

system into different parts and successively applying the foregoing solutions for series and for parallel circuits. This

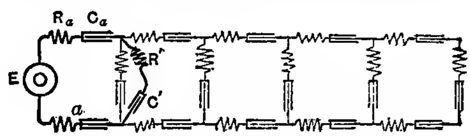


FIG. 101.—PROBLEM XXVII.

problem and its solution are similar to PROBLEM XII., and it will, therefore, suffice to merely outline the method to be

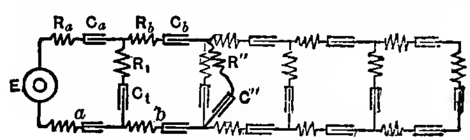


FIG. 102.—PROBLEM XXVII.

followed. The circuits 1, 2, 3, etc., have resistances and capacities R_1, R_2, R_3, etc., and C_1, C_2, C_3, etc. The resistance and capacities of the mains are R_a and C_a for the portion a; R_b and C_b for the portion b between circuits 1 and 2; R_c and C_c for the portion C; etc. R' and C' are the equivalent resistance and capacity for circuit 1 and the part of the system beyond, as indicated in Fig. 101. R'' and C'' are the equivalent resistance and capacity for circuit

2 and the part of the system beyond, as indicated in Fig. 102. Similarly, R''', R'''', C''', C'''' have values as indicated. The values for the equivalent resistances and capacities are found by the successive applications of the formulæ (356 *a*) and (356 *b*). The complete solution is given in Fig. 103, and its construction is similar throughout to

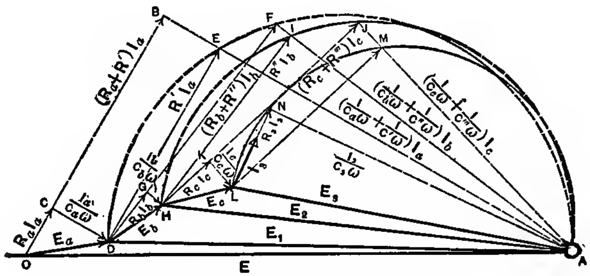


FIG. 103.—PROBLEM XXVII.

that of Fig. 76, PROBLEM XII. E_1, E_2, E_3, etc., give the impressed E. M. F.'s of the several parallel branches. By erecting an E. M. F. triangle on each, the effective E. M. F. and so the current in each branch may be found in the usual way. Thus in branch 3, $\overline{LN}$ is the effective E. M. F., and I_3 the current. E_a, E_b, E_c, etc., give the impressed E. M. F.'s in the portions *a*, *b*, *c*, etc., respectively, and the currents are easily found from the effective E. M. F.'s $R_a I_a$, $R_b I_b$, $R_c I_c$, etc. The full construction can best be followed by comparing PROBLEM XII., the similar case of circuits with resistance and self-induction.

CHAPTER XX.

CIRCUITS CONTAINING RESISTANCE, SELF-INDUCTION, AND CAPACITY.

IN the foregoing chapters the complete graphical solutions have been given for any combination of circuits in series and parallel when the circuits contain resistance and self-induction or when they contain resistance and capacity. In the first, the impressed E. M. F. of the source is equal to the E. M. F. necessary to overcome resistance plus the E. M. F. necessary to overcome the counter E. M. F. of self-induction; in the second, the impressed E. M. F. is

equal to the E. M. F. necessary to overcome the resistance plus the E. M. F. necessary to overcome that of the condenser. In each of these cases the three E. M. F.'s were represented by the three sides of a triangle.

Where a circuit contains resistance, self-induction, and capacity there are four E. M. F.'s to be considered. The impressed E. M. F. is equal to the sum of the E. M. F.'s necessary to overcome the resistance, the self-induction, and the condenser E. M. F., respectively.

The E. M. F. to overcome resistance is $R\,I$; that to overcome the self-induction is $L\,\omega\,I$ and is 90° ahead of the current; and that to overcome the E. M. F. of the condenser is $\frac{I}{C\,\omega}$ and is 90° behind the current. These may be drawn as the lines $\overline{O\,A}$, $\overline{A\,B}$, and $\overline{B\,C}$, respectively, in Fig. 104, and the geometrical or vector sum $\overline{O\,C}$ accordingly represents the impressed E. M. F.

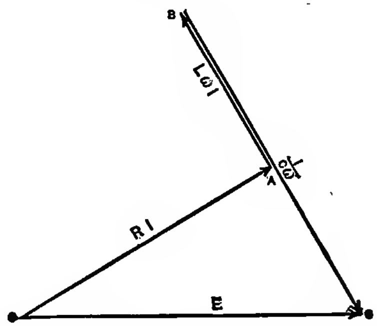


FIG. 104.—DIAGRAM OF ELECTROMOTIVE FORCES IN A CIRCUIT WITH RESISTANCE, SELF-INDUCTION, AND CAPACITY.

Now the E. M. F. to overcome that of self-induction and that of the condenser are always in exactly opposite directions, and when combined give one E. M. F. at right angles to the current. Thus, in Fig. 104, $\overline{A\,C}$ represents the combined effect of the E. M. F.'s $L\,\omega\,I$ and $\frac{I}{C\,\omega}$, represented by

$\overline{A\,B}$ and $\overline{B\,C}$, respectively, and is equal to $\left(\frac{1}{C\,\omega} - L\,\omega\right) I.$ We may, therefore, represent the E. M. F.'s in a circuit containing resistance, self-induction, and capacity by a triangle whose sides represent, respectively, the impressed E. M. F., that necessary to overcome resistance, and that necessary to overcome the E. M. F. of self-induction and capacity combined. Fig. 104 may then be drawn as Fig. 105. When $\frac{1}{C\,\omega}$ is greater than $L\,\omega$, the current is ahead of the impressed E. M. F.; and when $\frac{1}{C\,\omega}$ is less than $L\,\omega$, the current lags behind.

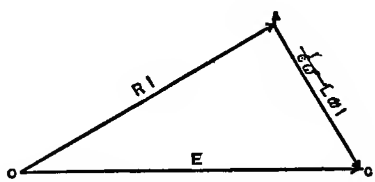


FIG. 105.—TRIANGLE OF ELECTROMOTIVE FORCES IN A CIRCUIT WITH RESISTANCE, SELF-INDUCTION, AND CAPACITY.

The tangent of this angle of lag or advance is

$$\tan\,\theta = \frac{\left(\frac{1}{C\,\omega} - L\,\omega\right) I}{R\,I} = \frac{1}{C\,R\,\omega} - \frac{L\,\omega}{R}.$$

When positive, the angle is one of advance; when negative, one of lag. Tan θ = reactance $\div$ resistance.

The impressed E. M. F. $\overline{O\,C}$, being equal to the square root of the sum of the squares of the two sides of the triangle, is

$$E = I\sqrt{R^2 + \left(\frac{1}{C\,\omega} - L\,\omega\right)^2}.$$

But this radical is the expression called the impediment (see page 131), and we may therefore write

$$\text{Current} = \frac{\text{E. M. F.}}{\text{Impediment}},$$

which corresponds to Ohm's law.

We have now established the graphical method of representing the E. M. F.'s in a simple circuit containing resistance, self-induction, and capacity, basing it upon the graphical solutions already given for circuits containing resistance and self-induction, and circuits containing resistance and capacity alone. These were separately obtained from the analytical equations previously given.

Let us now compare these graphical methods with the analytical results obtained in the discussion of circuits containing resistance, self-induction, and capacity. The general solution for current in a circuit with an harmonic impressed E. M. F. is [see (181)]

$$i = \frac{E}{\sqrt{R^2 + \left(\frac{1}{C\omega} - L\omega\right)^2}} \sin\left\{\omega t + \tan^{-1}\left(\frac{1}{CR\omega} - \frac{L\omega}{R}\right)\right\}.$$

This shows that the current has an angle of lag or advance whose tangent is

$$\frac{1}{CR\omega} - \frac{L\omega}{R},$$

the angle being advance when positive and lag when negative, which corresponds to the graphical construction just given. The maximum value of the current is

$$I = \frac{E}{\sqrt{R^2 + \left(\frac{1}{C\omega} - L\omega\right)^2}} = \frac{E}{\text{Impediment}}.$$

These equations, being identical with those just obtained graphically, show that the analytical results are correctly represented by this graphical method.

CAPACITY OR SELF-INDUCTION WHICH IS EQUIVALENT TO A COMBINATION OF SELF-INDUCTION AND CAPACITY.

Let C' or L' denote the capacity or self-induction which is equivalent to a given combination of the two, that is, which allows the same current to flow in the circuit when it is substituted for the combination. Referring to Fig. 105, we see that the E. M. F. of the combination is $\frac{I}{C\,\omega} - L\,\omega\,I$. Regarding this as a positive quantity, i.e., supposing $\frac{1}{C\,\omega} > L\,\omega$, we may put

$$\frac{I}{C'\omega} = \frac{I}{C\,\omega} - L\,\omega\,I; \text{ or } \frac{1}{C'\omega} = \frac{1}{C\,\omega} - L\,\omega.$$

(357) Hence $$C' = \frac{1}{\frac{1}{C} - L\,\omega^2} = \frac{C}{1 - L\,C\,\omega^2},$$

which is positive since $1 > L\,C\,\omega^2$.

If we suppose $\frac{1}{C\,\omega} < L\,\omega$, then $L\,\omega - \frac{1}{C\,\omega}$ is positive. We may then put

$$L'\,\omega I = L\,\omega\,I - \frac{I}{C\,\omega}; \quad \text{or} \quad L'\omega = L\,\omega - \frac{1}{C\,\omega}.$$

(358) Hence $L' = L - \frac{1}{C\,\omega^2}$, a positive quantity.

Problem XXVIII.—Effects of the Variation of the Constants *R*, *L*, *C*, and ω.

RESISTANCE VARIED.

If the resistance alone be varied in a circuit containing self-induction and capacity, it is interesting to inquire how

the current changes. Since the combination of the self-induction and capacity is equivalent to a self-induction or a capacity, we may substitute this equivalent for the combination. The change in the current caused by any variation in the resistance must therefore be the same as that before explained (PROBLEMS I. and XVII.) in the case of self-induction or capacity alone.

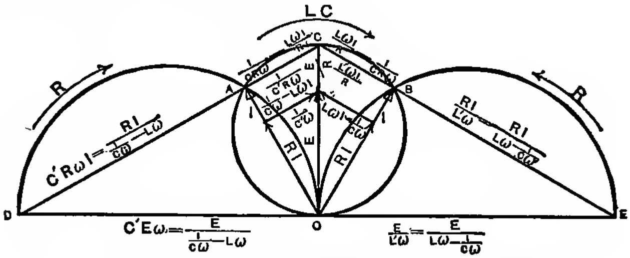


FIG. 106.—VARIATION OF CONSTANTS. PROBLEM XXVIII.

In Fig. 106, $\overline{O\,C}$ represents the impressed E. M. F., E, divided by the resistance R. The current I may either advance ahead of or lag behind $\overline{O\,C}$ according to whether $\frac{1}{C\,\omega}$ is greater or less than $L\,\omega$. For certain values of resistance, self-induction, and capacity let $\overline{O\,A}$ represent the current in advance of the impressed E. M. F., which signifies that $\frac{1}{C\,\omega} > L\,\omega$. Make $\overline{D\,O}$ equal to $C'\,E\,\omega$, and draw the semi-circle $O\,A\,D$ upon $\overline{D\,O}$ as diameter. This is then the locus of the current vector $\overline{O\,A}$ as the resistance alone changes, as explained in Problem XVII.

Similarly, if the quantity $\frac{1}{C\,\omega} - L\,\omega$ had been negative and of an equal magnitude to its former positive value, the current would have been represented by $\overline{O\,B}$ lagging behind $\overline{O\,C}$, and any variation in the resistance alone would cause the current vector to move upon the semi-circle $O\,B\,E$, being equal to $\overline{O\,E}$ when the resistance is zero, as explained

in PROBLEM I. It is to be noticed that Fig. 106 is the same as Figs. 52 and 86 combined.

The arrows R, R, show the direction of change as the resistance increases.

SELF-INDUCTION OR CAPACITY VARIED.

When either the self-induction or capacity alone is varied, it is evident that the value of the quantity $\frac{1}{C\,\omega} - L\,\omega$ and, therefore, the value of the equivalent self-induction, L', or equivalent capacity C', is changed. Now any variation in the equivalent self-induction will cause the current vector to move on the semi-circle $O\,B\,C$, as explained in PROBLEM I., and any variation in the equivalent capacity will cause the current vector to move on the semi-circle $O\,A\,C$, as explained in PROBLEM XVII. Any change, then, in self-induction or capacity will cause the current to move through some part of the circle $O\,A\,C\,B$, whose diameter is $\overline{O\,C}$ equal to $\frac{E}{R}$.

The arrow L, C shows the direction of change as the capacity or the self-induction increases.

FREQUENCY VARIED.

When the frequency of alternation is varied, it is equivalent to a variation of ω, the angular velocity, which is equal to $2\,\pi$ times the frequency. Any increase in the frequency increases the effect of the self-induction or the capacity. If the self-induction is the more important element and the circuit has an equivalent self-induction [see equation (358)],

$$L' = L - \frac{1}{C\,\omega^2},$$

any variation in the frequency will cause a variation in the equivalent self-induction according to this equation. If the capacity is the more important element, the equivalent capacity varies with ω according to the equation (357),

$$C' = \frac{C}{1 - L C \omega^2}.$$

It has just been shown that any variation in the equivalent self-induction or capacity causes the current vector to move, between limits, on the circle $O A C B$. This, then, is the effect of a change in frequency. The direction of this change, as the frequency increases, is shown by the arrow L, C in Fig. 106.

Problem XXIX.—Series Circuit. Current Given.

Let there be a circuit having n different coils and condensers in series as represented in Fig. 107. It is required to find the impressed E. M. F. necessary to cause the current I to flow, and the difference of potential at the terminals of each coil and condenser.

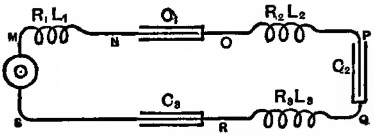


FIG. 107.—PROBLEM XXIX. AND PROBLEM XXX.

In Fig. 108 make $\overline{OA}$ equal to the given current I. Lay off $\overline{OB}$ equal to $R_1 I_1$, and make $\overline{BC}$ equal to $L_1 \omega I$ perpendicular to $\overline{OB}$ in the positive direction. Then in the negative direction make $\overline{BD}$ equal to $\dfrac{I}{C_1 \omega}$. The algebraic sum of $\overline{BC}$ and $\overline{BD}$ is $\overline{BE}$. Then $\overline{OE}$ represents the potential difference between M and O, Fig. 107;

$\overline{OC}$, the potential difference between M and N; and $\overline{BD}$, or $\overline{CE}$, the potential difference between N and O, the terminals of the condenser. In a similar manner the lines $\overline{EG}$, $\overline{EI}$, $\overline{IK}$, *and* $\overline{IM}$ are drawn representing the poten-

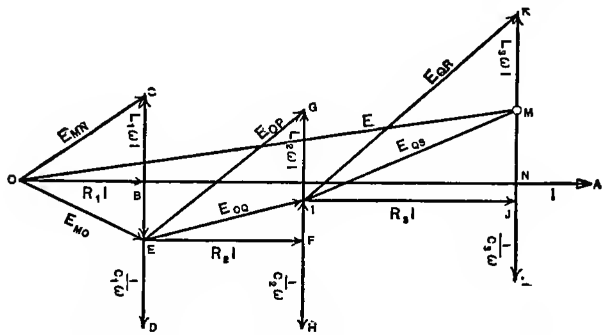


FIG. 108.—PROBLEM XXIX AND PROBLEM XXX.

tial difference between OP, OQ, QR, and QS, respectively, Fig. 107, until we finally reach the point M, Fig. 108. $\overline{OM}$ is then the required impressed E. M. F. necessary to cause the current I to flow.

EQUIVALENT RESISTANCE, SELF-INDUCTION, OR CAPACITY OF SERIES CIRCUITS.

It is evident from Fig. 108 that if we had one coil only, whose self-induction L' is such that the line $\overline{NM}$ is equal to $L' \omega I$, and whose resistance R' is such that the line $\overline{ON}$ is equal to $R' I$, the same current I would flow if this coil be substituted for the combination of condensers and coils. The resistance of this equivalent coil must evidently be

$$(359) \qquad R' = R_1 + R_2 + R_3 + \text{etc.} = \Sigma R.$$

The self-induction of the coil, being represented by $\overline{MN}$ divided by ωI, is evidently found thus :

$$\overline{MN} = \overline{BC} - \overline{BD} + \overline{FG} - \overline{FH} + \overline{JK} - JL, \text{ or}$$

$$L' \omega I = L_1 \omega I - \frac{I}{C_1 \omega} + L_2 \omega I - \frac{I}{C_2 \omega} + L_3 \omega I - \frac{I}{C_3 \omega}.$$

$$L' \omega I = I \sum \left(L \omega - \frac{1}{C \omega} \right).$$

$$(360) \qquad L' = \frac{1}{\omega} \sum \left(L \omega - \frac{1}{C \omega} \right).$$

If it happens that this sum is a negative quantity, the self-induction cannot replace the combination, but a condenser can. It will be seen that the capacity of this condenser C' may be found as follows:

$$\frac{I}{C' \omega} = \sum \left(\frac{I}{C \omega} - L \omega I \right) = I \sum \left(\frac{1}{C \omega} - L \omega \right).$$

$$(361) \qquad \text{Hence} \qquad C' = \frac{1}{\omega \sum \left(\frac{1}{C \omega} - L \omega \right)}.$$

These equations, (359), (360), and (361), give the means for computing the equivalent resistance, self-induction, and capacity of series circuits.

Problem XXX.—Series Circuit. Impressed E. M. F. Given.

Suppose the impressed E. M. F., represented by the line $\overline{OM}$, Fig. 108, is given, and the circuit is that shown in Fig. 107. It is required to find what current will flow and what is the E. M. F. at the terminals of each coil and condenser.

If the equivalent self-induction L' given by equation (360) above, or the equivalent capacity C' given by equation (361), is calculated, we may construct the triangle of E. M. F.'s

OMN, in which $\overline{ON}$ equals $I\Sigma R$, and $\overline{NM}$ equals $I\Sigma\left(L\omega - \frac{1}{C\omega}\right)$. The current $\overline{OA}$ is found by dividing $\overline{ON}$ by ΣR. After we have obtained the value of the current, we may proceed, as in the preceding problem, to find the E. M. F. in each part of the circuit.

Problem XXXI.—Divided Circuit. Impressed E. M. F. Given.

Let us consider the problem of a divided circuit having resistance, self-induction, and capacity in each branch, as shown in Fig. 109. The impressed E. M. F., E, is given;

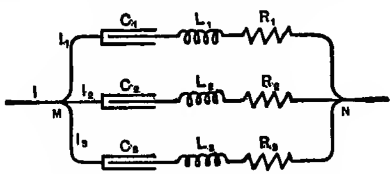


FIG. 109.—PROBLEM XXXI.

it is required to find the main and branch currents. The construction in Fig. 110 gives the complete solution. Since the impressed E. M. F. at the terminals of each branch is known, each may be separately treated as a simple circuit containing resistance, self-induction, and capacity, as in PROBLEM XXX. Upon $\overline{OA}$, which represents the impressed E. M. F., E, a circle is drawn, and upon $\overline{OA}$ the several E. M. F. triangles OBA, OCA, ODA, are erected with angles θ_1, θ_2, θ_3, of advance or lag according as $\frac{1}{C\omega}$ is greater or less than $L\omega$. The currents I_1, I_2, I_3 are found by dividing the corresponding effective E. M. F.'s by the resistance R_1, R_2, R_3, respectively, and the main current, I, is found by taking the vector sum of the branch currents. The problem is in every way the same as the

problem of the parallel circuits with the resistance and self-induction, or with resistance and capacity, except that the current in any one branch may be either in advance or behind the impressed E. M. F., according to the particular values of the resistance, self-induction, and capacity of that branch.

Equivalent Resistance, Self-induction, and Capacity of Parallel Circuits.

Let us suppose that for the parallel system there is substituted a simple circuit containing resistance and self-induction, or resistance and capacity, such that the same main current will flow. The investigation of the values of equivalent resistance, self-induction, and capacity is similar

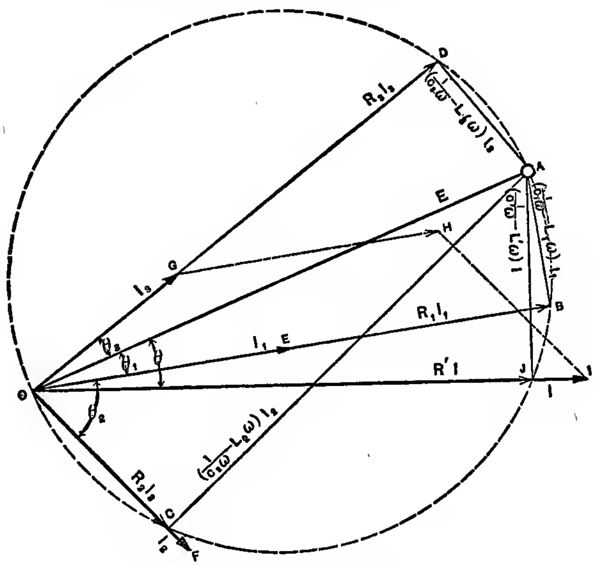


Fig. 110.—Problem XXXI.

to the determination of equivalent resistance and self-induction, Problem V., and first appeared in a paper by

the authors in the *Philosophical Magazine* for September, 1892.

If we take the projections of the currents I, I_1, I_2, etc., upon the line $\overline{OA}$, we obtain the equation

$$(362)\qquad I\cos\theta = I_1\cos\theta_1 + I_2\cos\theta_2 + \ldots = \Sigma I\cos\theta.$$

If we consider the projections of the currents upon a line perpendicular to $\overline{OA}$, we obtain

$$(363)\qquad I\sin\theta = I_1\sin\theta_1 + I_2\sin\theta_2 + \ldots = \Sigma I\sin\theta.$$

Since all the triangles $O\,B\,A$, $O\,C\,A$, etc., are right triangles, we get the following relations:

$$(364)\qquad I = \frac{E}{\sqrt{R'^2 + \left(\frac{1}{C'\omega} - L'\omega\right)^2}},$$

$$I_1 = \frac{E}{\sqrt{R_1^2 + \left(\frac{1}{C_1\omega} - L_1\omega\right)^2}},$$

$$I_2 = \frac{E}{\sqrt{R_2^2 + \left(\frac{1}{C_2\omega} - L_2\omega\right)^2}}, \text{ etc.}$$

$$(365)\qquad \cos\theta = \frac{R'}{\sqrt{R'^2 + \left(\frac{1}{C'\omega} - L'\omega\right)^2}},$$

$$\cos\theta_1 = \frac{R_1}{\sqrt{R_1^2 + \left(\frac{1}{C_1\omega} - L_1\omega\right)^2}},$$

$$\cos\theta_2 = \frac{R_2}{\sqrt{R_2^2 + \left(\frac{1}{C_2\omega} - L_2\omega\right)^2}}, \text{ etc.}$$

$$\text{(366)} \qquad \sin\theta = \frac{\dfrac{1}{C'\omega} - L'\omega}{\sqrt{R'^2 + \left(\dfrac{1}{C'\omega} - L'\omega\right)^2}},$$

$$\sin\theta_1 = \frac{\dfrac{1}{C_1\omega} - L_1\omega}{\sqrt{R_1^2 + \left(\dfrac{1}{C_1\omega} - L_1\omega\right)^2}},$$

$$\sin\theta_2 = \frac{\dfrac{1}{C_2\omega} - L_2\omega}{\sqrt{R_2^2 + \left(\dfrac{1}{C_2\omega} - L_2\omega\right)^2}}, \text{ etc.}$$

Substituting these values in (362), we have

$$\text{(367)} \qquad \frac{I\cos\theta}{E} = \frac{R'}{R'^2 + \left(\dfrac{1}{C'\omega} - L'\omega\right)^2}$$

$$= \sum \frac{R}{R^2 + \left(\dfrac{1}{C\omega} - L\omega\right)^2} = A.$$

Making a similar substitution in (363), we have

$$\text{(368)} \qquad \frac{I\sin\theta}{E} = \frac{\dfrac{1}{C'\omega} - L'\omega}{R'^2 + \left(\dfrac{1}{C'\omega} - L'\omega\right)^2}$$

$$= \sum \frac{\dfrac{1}{C\omega} - L\omega}{R^2 + \left(\dfrac{1}{C\omega} - L\omega\right)^2}$$

$$= \sum \frac{C - C^2 L\omega^2}{C^2 R^2 \omega^2 + (1 - C L\omega^2)^2}\,\omega = B\,\omega.$$

Here the letters A and B are introduced to simplify the resulting expressions.

Dividing (368) by (367), we have

$$(369) \qquad \tan\theta = \frac{B\,\omega}{A}.$$

Comparing (365) and (367), we obtain

$$(370) \qquad A = \frac{\cos^2\theta}{R'}, \quad \text{or} \quad R' = \frac{\cos^2\theta}{A}.$$

Comparing (366) and (368), we obtain

$$(371) \quad B\,\omega = \frac{\sin^2\theta}{\dfrac{1}{C'\,\omega} - L'\,\omega}, \quad \text{or} \quad \frac{1}{C'\,\omega} - L'\,\omega = \frac{\sin^2\theta}{B\,\omega}.$$

For $\cos^2\theta$ and $\sin^2\theta$ we may substitute the values

$$\cos^2\theta = \frac{1}{1+\tan^2\theta} = \frac{1}{1+\dfrac{B^2\,\omega^2}{A^2}} = \frac{A^2}{A^2+B^2\,\omega^2},$$

$$\sin^2\theta = \frac{1}{1+\cot^2\theta} = \frac{1}{1+\dfrac{A^2}{B^2\,\omega^2}} = \frac{B^2\,\omega^2}{A^2+B^2\,\omega^2}.$$

With these substitutions equations (370) and (371) become

$$(372) \qquad R' = \frac{A}{A^2+B^2\,\omega^2},$$

$$(373) \qquad \frac{1}{C'\,\omega} - L'\,\omega = \frac{B\,\omega}{A^2+B^2\,\omega^2}.$$

Here A and $B\,\omega$ each stand for a summation, as expressed in (367) and (368), and are calculated from the particular values of the resistance, self-induction, and capacity of each branch. This gives a definite value to the equivalent resistance, R', according to (372), and a de-

finite value to $\frac{1}{C' \omega} - L' \omega$, according to (373). There may be an indefinite number of values assigned to L' or C' according to values assigned to the other, that is, we may assume any value for L' and by (373) determine the value for C', or vice versa.

If the right-hand member of (373) is positive, we may consider that the equivalent circuit has no self-induction, i.e., $L' = 0$, and calculate the equivalent capacity. If this member is negative, we may consider that the equivalent circuit has no condenser, i.e., $C' = \infty$, and calculate accordingly the equivalent self-induction. In any case, therefore, we may speak of the equivalent resistance and self-induction, or the equivalent resistance and capacity of a combination of circuits, according as the equivalent simple circuit would have self-induction or capacity.

The angle of lag or advance of the main current is obtained from equation (369).

Branch Circuits with Resistance and Self-induction only.

There is no condenser in any branch and the capacity of each is, therefore, infinite. We can, accordingly, obtain the expressions for A and $B\,\omega$ for this case by substituting $C = \infty$ in the summations in (367) and (368). This gives

$$A = \sum \frac{R}{R^2 + L^2\,\omega^2}.$$

$$-B\,\omega = \sum \frac{L\,\omega}{R^2 + L^2\,\omega^2}.$$

From (372) and (373), we have

$$R' = \frac{A}{A^2 + B^2\,\omega^2}.$$

$$L'\,\omega = \frac{-B\,\omega}{A^2 + B^2\,\omega^2}.$$

These results are seen to be identical with those obtained in Problem V. and given in equations (352) and (353).

Branch Circuits with Resistance and Capacity only.

In this case there is no self-induction in any branch, and the expressions for A and $B\,\omega$ are found by substituting $L = 0$ in the summations in (367) and (368). This gives

$$A = \sum \frac{R}{R^2 + \dfrac{1}{C\,\omega^2}}.$$

$$B\,\omega = \sum \frac{\dfrac{1}{C\,\omega}}{R^2 + \dfrac{1}{C^2\,\omega^2}} = \sum \frac{C\,\omega}{C^2\,R^2\,\omega^2 + 1}.$$

The expression for R' is the same as that in (372), and from (373) we get an expression for the equivalent capacity, thus:

$$\frac{1}{C'\,\omega} = \frac{B\,\omega}{A^2 + B^2\,\omega^2}.$$

These results are identical with those previously given in Problem XXI.

Problem XXXII.—Example of a Divided Circuit, Impressed E. M. F. Given.

Suppose a divided circuit has a condenser with a capacity C of one micro-farad in one branch, and a coil whose self-induction L is one henry and resistance R one

hundred ohms in the other branch, as in Fig. 111. Let the impressed E. M. F. be one thousand volts, and 2π times

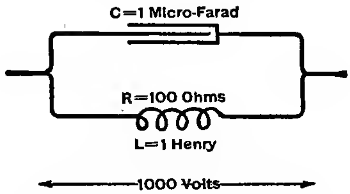


FIG. 111.—PROBLEM XXXII.

the frequency be one thousand. What are the currents in the main line and branches?

Since there is no resistance in the condenser branch, the current, $\overline{OB}$, Fig. 112, is ninety degrees in advance of

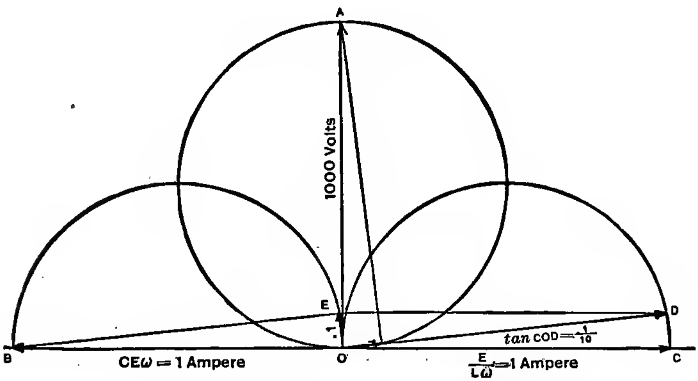


FIG. 112.—PROBLEM XXXII.

the impressed E. M. F., $\overline{OA}$, and is equal to $CE\omega = 10^{-6} \times 1000 \times 1000 = 1$ ampere. The tangent of the angle of lag of the current in the coil is $\frac{L\omega}{R}$, equal to $\frac{1 \times 1000}{100} = 10$, and therefore the current, $\overline{OD}$, in the coil is almost ninety degrees behind the impressed E. M. F.

The current $\overline{OD}$ is almost equal to one ampere, for $\overline{OD}$ is almost equal to $\overline{OC}$, and

$$\overline{OC} = \frac{E}{L\omega} = \frac{1000}{1 \times 1000} = 1 \text{ ampere.}$$

We have, then, the condenser current $\overline{OB}$ and the coil current $\overline{OD}$, each equal approximately to one ampere, one in advance of the E. M. F. and the other lagging behind. The resultant of these two branch currents is $\overline{OE}$ and is equal to one tenth of an ampere, approximately; that is, each branch current is about ten times as large as the main current. In this case the main current is almost in phase with the impressed E. M. F., being in advance of it by a small angle.

Problem XXXIII. Divided Circuit. Current Given.

If we have a number of parallel circuits, containing resistance, self-induction, and capacity, and know the value of the main current, I, the solution is similar to that given in PROBLEM VI. The first method of solution consists in assuming an impressed E. M. F., solving as in the previous problem, and then correcting the scale to agree with the given value of the current. The second method consists in computing the equivalent resistance and equivalent self-induction or capacity of the parallel system, according to the formulæ (372) and (373), finding graphically the impressed E. M. F., and then solving according to the last problem.

Problem XXXIV.—Series and Parallel Combinations of Circuits.

In the graphical treatment of circuits with resistance and self-induction, and of circuits with resistance and capacity, the discussion was given first of series circuits

and then of circuits connected in parallel. It was then shown how problems arising from any combination of circuits in series and parallel could be readily solved by repeated applications of the methods given for the solution of series and parallel circuits. In the problems given for circuits containing resistance, self-induction, and capacity the full solution has been given for series and for parallel circuits. These principles may be applied in solving any combination of series and parallel circuits, and to go through particular examples of these would be needless. The same problems as those given for a circuit with resistance and self-induction or capacity can be solved in the same way if the circuits contain all three. The problems given have been selected as examples and not as exhaustively representing all the problems which these graphical methods are adapted to solve. The various combinations which arise are endless and may often be solved in more ways than one. The choice of method depends upon the particular requirements of the problem. A clear idea of the principles involved in the simple cases will enable one to extend them with ease to whatever problems arise.

APPENDIX A.

RELATION BETWEEN PRACTICAL AND C. G. S. UNITS.

ELECTRICAL UNITS.

	Practical System.	C. G. S. System. Electro-magnetic.	C. G. S. System. Electrostatic.
Quantity	1 coulomb	10^{-1}	$v \times 10^{-1} = 3 \times 10^{9}$
Current	1 ampere..	10^{-1}	$v \times 10^{-1} = 3 \times 10^{9}$
Potential	1 volt	10^{8}	$10^{8} \div v = \frac{1}{3} \times 10^{-2}$
Resistance	1 ohm	10^{9}	....
Capacity	1 farad...	10^{-9}	$v^{2} \times 10^{-9} = 9 \times 10^{11}$
Self-induction	1 henry...	10^{9}	....
Mutual induction.	1 henry...	10^{9}	....

(v = velocity of light $= 3 \times 10^{10}$.)

MECHANICAL UNITS.

Practical System.	C. G. S. System.
Unit length	$= 10^{9}$ cm.
Unit mass	$= 10^{-11}$ grms.
Unit time	$= 1$ sec.
One joule	$= 10^{7}$ ergs.
One watt $= \frac{1}{746}$ h. p.	$= 10^{7}$ ergs per sec.

APPENDIX B.

SOME MECHANICAL AND ELECTRICAL ANALOGIES.

TABLE I.—LINEAR MOTION.

Notation.

1. Time $= t$.
2. Distance $= s$.
3. Linear velocity $= v = \frac{d\,s}{d\,t}$; or, $d\,s = v\,d\,t$.
4. Linear acceleration $= a = \frac{d\,v}{d\,t} = \frac{d^2 s}{d\,t^2}$.
5. Mass $= M$.
6. Momentum $= Mv$.

Frictional Resistance.

7. Frictional resistance $= R$.
8. Force to overcome resistance $= F_R = R\,v$.
9. Energy expended in the time $d\,t$ in overcoming resistance $= d\,W_R = F_R\,d\,s = R\,v^2\,d\,t$.

Inertia.

10. Force to overcome inertia $= F' = M\,a = M\frac{d\,v}{d\,t}$.
11. Kinetic energy acquired in the time $d\,t$

$$= d\,W' = F'\,d\,s = M\,v\frac{d\,v}{d\,t}\,d\,t.$$

12. Kinetic energy $= W' = \int_0^v M\,v\frac{d\,v}{d\,t}\,d\,t = \frac{1}{2}M\,v^2$.

Resistance plus Inertia.

13. Total force applied $= F = F_R + F' = R\,v + M\frac{d\,v}{d\,t}$.
14. Total energy supplied in the time $d\,t$

$$= d\,W = d\,W_R + d\,W'; \quad \text{or,} \quad F\,d\,s = F_R\,d\,s + F'\,d\,s;$$

$$\text{or,} \quad F\,v\,d\,t = R\,v^2\,d\,t + M\,v\frac{d\,v}{d\,t}\,d\,t.$$

TABLE II. ROTARY MOTION.

Notation.

1. Time $= t$.
2. Angle $= \phi$.
3. Angular velocity $= \omega = \frac{d\phi}{dt}$; or, $d\phi = \omega\, dt$.
4. Angular acceleration $= \alpha = \frac{d\omega}{dt} = \frac{d^2\phi}{dt^2}$.
5. Moment of inertia $= I$.
6. Angular momentum $= I\omega$.

Frictional Resistance.

7. Frictional resistance $= R$.
8. Torque to overcome resistance $= T_R = R\,\omega$.
9. Energy expended in the time dt in overcoming resistance $= dW_R = T_R\, d\phi = R\,\omega^2\, dt$.

Inertia.

10. Torque to overcome inertia $= T' = Ia = I\frac{d\omega}{dt}$.
11. Kinetic energy acquired in the time dt

$$= dW' = T'\, d\phi = I\omega\frac{d\omega}{dt}\, dt.$$

12. Kinetic energy $= W' = \int_0^\omega I\omega \frac{d\omega}{dt}\, dt = \frac{1}{2} I\omega^2$.

Resistance plus Inertia.

13. Total torque applied $= T = T_R + T' = R\,\omega + I\frac{d\omega}{dt}$.
14. Total energy supplied in the time dt

$$= dW = dW_R + dW'; \text{ or, } \quad T\,d\phi = T_R\, d\phi + T'\, d\phi;$$

$$\text{or, } \quad T\omega\, dt = R\,\omega^2\, dt + I\omega\frac{d\omega}{dt}\, dt.$$

TABLE III.—ELECTRIC CURRENT.

Notation.

1. Time $= t$.
2. Quantity $= q$.
3. Current $= i = \frac{dq}{dt}$; or, $dq = i\,dt$.
4. Current acceleration $= \beta = \frac{di}{dt}$.
5. Coefficient of self-induction $= L$.
6. Electro-magnetic momentum $= Li$.

Ohmic Resistance.

7. Ohmic resistance $= R$.
8. Electromotive force to overcome resistance $= e_R = Ri$.
9. Energy expended in the time dt in overcoming resistance $= dW_R = e_R\,dq = Ri^2\,dt$.

Self-induction.

10. Electromotive force to overcome self-induction

$$= e' = L\beta = L\frac{di}{dt}.$$

11. Energy acquired by the magnetic field in the time dt

$$= dW' = e'\,dq = Li\frac{di}{dt}dt.$$

12. Energy of magnetic field $= W' = \int_0^I Li\frac{di}{dt}\,dt = \tfrac{1}{2}LI^2$.

Resistance plus Self-induction.

13. Total electromotive force applied

$$= e = e_R + e' = Ri + L\frac{di}{dt}.$$

14. Total energy supplied in the time dt

$$= dW = dW_R + dW'; \quad \text{or,} \quad e\,dq = e_R\,dq + e'\,dq;$$

$$\text{or,} \quad ei\,dt = Ri^2\,dt + Li\frac{di}{dt}\,dt.$$

APPENDIX C.

NOTATION USED THROUGHOUT THIS BOOK.

(Numbers refer to page where first used.)

A. Area, 67; or, constant, 41.
B. Constant, 41.
B. Induction per square centimeter, 22.
C. Capacity, 64; or, constant, 41.
C'. Equivalent capacity, 279.
D. Symbolic operator, 84.
E. Constant E. M. F., 25; or, maximum value of harmonic E. M. F., 50.
$\bar{E}$. Virtual E. M. F., i.e., square root of mean square value, 38 and 143.
F. Force, 20.
H. Magnetizing force, 21.
I. Constant current, 25; or, maximum value of harmonic current, 53.
$\bar{I}$. Virtual current, i.e., square root of mean square value, 38 and 143.
Im. Impedance, 188.
L. Coefficient of self-induction, 23.
L'. Equivalent self-induction, 235.
N. Total induction, i.e., total number of lines, 21.
O. Origin. Center of revolution, 33.
Q. Constant quantity; or, charge of electricity, 25.
R. Resistance, 24.
R'. Equivalent resistance, 235.

T. Period, 33; or, time constant, 46.
V. Potential, 63.
W. Work or energy, 28.

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a. Amplitude, 33; or, constant, 86.
b. Constant, 57.
c. Arbitrary constant of integration, 44.
d. Distance, 67.
e. Instantaneous value of electromotive force, 25.
f. Arbitrary function, 43.
f'. First differential coefficient of f, 71.
h. Constant, 184.
i. Instantaneous value of current, 25.
j. $\sqrt{-1}$, 93.
k. Constant, 183.
l. Constant length, 201.
m. Strength of pole, 20; or, constant, 96.
n. Frequency, 34; or, constant, 58.
p. An abbreviation, 191.
q. Instantaneous value of charge, 25.
r. Distance, 20; or, constant, 190.
t. Time, 34.
x. Independent variable, 41; also length or distance, 178.
y. Dependent variable, 34.
z. Dependent variable, 42.

α. An abbreviation, 191; or, a constant, 41.
β. Constant, 41.
γ. Constant, 41.
ϵ. Naperian base, (2.71828), 44.
$+\theta$. Angle, usually of advance, 35.
$-\theta$. Angle, usually of lag, 35.

κ. Specific inductive capacity, 61; or, Constant, 206.

λ. Wave-length, 196.

μ. Permeability, 22.

π. Ratio of circumference to diameter, (3.14159), 21.

Σ. Summation, 59.

τ. $1 \div$ time-constant, $\frac{1}{T}$, 126.

Φ. Arbitrary constant, 95.

ϕ. Angle, 34.

χ. Angle, 150.

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ω. Angular velocity, $2\pi n$, 34.

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